INVARANTS RELATED TO THE BERGMAN KERNEL OF A BOUNDED DOMAIN IN C^n

TADAYOSHI KANEMARU

ABSTRACT. In this paper we introduce biholomorphic invariants using the Bergman kernel function of a bounded domain in C^n.

Let $K_D(z,\bar{t})$ be the Bergman kernel function of a bounded domain $D$ in C^n. As is well known, $K_D(z,\bar{t})$ admits the following transformation rule [1]:

Let $D, \Delta$ be bounded domains and $w = w(z)$ a biholomorphic mapping from $D$ onto $\Delta$. Then

\begin{equation}
K_D(z,\bar{t}) = \det \frac{d\tau}{dt} K_\Delta(w,\bar{\tau}) \det \frac{dw}{dz} \quad (\tau = w(t)).
\end{equation}

Moreover,

\begin{equation}
T_D(z,\bar{t}) = \frac{\partial^2}{\partial t^* \partial z} \log K_D(z,\bar{t}),
\end{equation}

which is defined when $K_D(z,\bar{t}) \neq 0$ and is uniquely determined by $D$, is a relative invariant under biholomorphic mappings, that is,

\begin{equation}
T_D(z,\bar{t}) = \left( \frac{d\tau}{dt} \right)^* T_\Delta(w,\bar{\tau}) \frac{dw}{dz}.
\end{equation}

In particular, the Bergman metric

\[ ds^2 = dz^* T_D(z,\bar{z}) \, dz \]

is invariant under biholomorphic mappings.

Throughout this paper we use the following notation: $z = (z_1, z_2, \ldots, z_n)'$, $w = w(z) = (w_1(z), w_2(z), \ldots, w_n(z))'$,

\[ \frac{\partial}{\partial z} = \left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n} \right), \quad \frac{dw}{dz} = \frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \times w, \]

where the symbols $'$, $^*$ and $\times$ stand for transposition, conjugated transposition and Kronecker product, respectively.

The above invariants make it possible to introduce some other biholomorphic invariants.

We define

\[ K_{D,(p,q)}(z,\bar{t}) = K_P^D(z,\bar{t})(\det T_D(z,\bar{t}))^q \quad (p, q \geq 0), \]

\[ T_{D,(p,q)}(z,\bar{t}) = \frac{\partial^2}{\partial t^* \partial z} \log K_{D,(p,q)}(z,\bar{t}). \]
Then we have the following formulas [3]:

\[(3) \quad K_{D,(p,q)}(z,\bar{t}) = \left( \det \frac{d\tau}{dt} \right)^{p+q} K_{\Delta,(p,q)}(w,\bar{\tau}) \left( \det \frac{dw}{dz} \right)^{p+q}, \]

\[(4) \quad T_{D,(p,q)}(z,\bar{t}) = \left( \frac{d\tau}{dt} \right)^* T_{\Delta,(p,q)}(w,\bar{\tau}) \left( \frac{dw}{dz} \right). \]

In particular,

\[d\sigma_D^2 = d\sigma_{D,(p,q)}^2 = dz^* T_{D,(p,q)}(z,\bar{z}) dz \]

is a Kähler metric which is invariant under biholomorphic mappings.

We note that our metric \(d\sigma^2\) gives the Bergman metric for \(p = 1, q = 0\), and the Burbea metric for \(p = n + 1, q = 1\) [2, 3].

Making use of (1) and (4), it can be shown that

\[J_{D,(p,q)}(z,\bar{z}) = \frac{\det T_{D,(p,q)}(z,\bar{z})}{K_D(z,\bar{z})} \]

is a positive biholomorphic invariant.

Similarly, from (3) we also deduce that

\[H_{D,(p,q)}(z,\bar{t}) = \frac{K_{D,(p,q)}(z,\bar{t}) K_{D,(p,q)}(t,\bar{z})}{K_{D,(p,q)}(t,\bar{t}) K_{D,(p,q)}(z,\bar{z})} \]

is a positive biholomorphic invariant. This extends the result in [5] for the special case of \(p = 1\) and \(q = 0\).

We shall now define \(R(z,\bar{t})\) by

\[R(z,\bar{t}) = \sqrt{\det T_{D,(p,q)}(t,\bar{t})} |\det SU(z,\bar{t})|, \]

where

\[U(z,\bar{t}) = T_{\Delta,(p,q)}^{-1}(t,\bar{t}) \int_t^z T_{D,(p,q)}(z,\bar{t}) dz, \]

and \(S\) is a vector differential operator such that

\[S = \left( \frac{d}{d\sigma}, \frac{d^2}{d\sigma^2}, \ldots, \frac{d^n}{d\sigma^n} \right), \quad d\sigma = ds_{D,(p,q)} = \sqrt{dz^* T_{D,(p,q)}(z,\bar{z}) dz}. \]

Then we have the following

**THEOREM.** \(R(z,\bar{t})\) is a nonnegative biholomorphic invariant.

**PROOF.** Let \(D, \Delta\) be bounded domains and \(w = w(z)\) be a biholomorphic mapping from \(D\) onto \(\Delta\). Then from (4) we have

\[U(z,\bar{t}) = T_{\Delta,(p,q)}^{-1}(t,\bar{t}) \int_t^z T_{D,(p,q)}(z,\bar{t}) dz \]

\[= \left( \frac{d\tau}{dt} \right)^{-1} T_{\Delta,(p,q)}^{-1}(\tau,\bar{\tau}) \left( \frac{d\tau}{dt} \right)^* \int_\tau^w \left( \frac{d\tau}{dt} \right)^* T_{\Delta,(p,q)}(w,\bar{\tau}) \frac{dw}{dz} \frac{dz}{dw} d\tau \]

\[= \left( \frac{d\tau}{dt} \right)^{-1} T_{\Delta,(p,q)}^{-1}(\tau,\bar{\tau}) \int_\tau^w T_{\Delta,(p,q)}(w,\bar{\tau}) dw, \]
where \( \tau = w(t) \). Thus

\[
U(z, \bar{t}) = (d\tau/dt)^{-1} U(w, \bar{\tau}).
\]

Differentiating this equation with respect to the metric \( d\sigma \), we obtain

\[
U^{(n)}(z, \bar{t}) \equiv \frac{d^n}{d\sigma^n} U(z, \bar{t}) = \left( \frac{d\tau}{dt} \right)^{-1} U^{(n)}(w, \bar{\tau}).
\]

It follows from (4) and (5) that

\[
R(z, \bar{t}) = \sqrt{\det T_{\Delta,(p,q)}(t, \bar{t}) | \det SU(z, \bar{t}) |}
\]

\[
= \sqrt{\det \left( \left( \frac{d\tau}{dt} \right)^* T_{\Delta,(p,q)}(\tau, \bar{\tau}) \left( \frac{d\tau}{dt} \right) \right) \det \left( \left( \frac{d\tau}{dt} \right)^{-1} SU(w, \bar{\tau}) \right)}
\]

\[
= \sqrt{\det T_{\Delta,(p,q)}(\tau, \bar{\tau}) | \det SU(w, \bar{\tau}) |}
\]

\[
= R(w, \bar{\tau}).
\]

This concludes the proof.

REMARK. This result agrees with that in [4] when \( p = 1 \) and \( q = 0 \).

REFERENCES


DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, KUMAMOTO UNIVERSITY, KUMAMOTO 860, JAPAN