A WAVE EQUATION
WITH A POSSIBLY JUMPING NONLINEARITY

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ABSTRACT. Existence of a doubly periodic solution to a forced semilinear wave equation is established. The nonlinearity may “jump” across any finite number of eigenvalues of finite multiplicity.

1. Introduction. Let \( J = [0, 2\pi] \times [0, 2\pi] \) and let \( f: J \times \mathbb{R} \to \mathbb{R}, (t, x, s) \mapsto f(t, x, s) \), be a function satisfying the Carathéodory conditions. Assume there is a number \( A > 0 \) and a function \( B \in L^2(J) \) such that for each \( s \in \mathbb{R} \) and \((t, x) \in J\) we have

\[
|f(t, x, s)| \leq A|s| + B(t, x).
\]

Let \( h \in L^2(J) \). We consider the existence (in the weak sense) of solutions \( 2\pi \)-periodic in each of \( x \) and \( t \) for the semilinear wave equation

\[
utt - uxx - f(t, x, u) = h(t, x).
\]

By a weak solution to the doubly \( 2\pi \)-periodic problem for (1.2) is meant a \( u \in L^2(J) \) such that

\[
\int_J u(t, x)[v_{tt}(t, x) - v_{xx}(t, x)] \, dt \, dx = \int_J [f(t, x, u(t, x)) + h(t, x)]v(t, x) \, dt \, dx
\]

for every \( v \in C^2(J) \) satisfying the boundary conditions

\[
\begin{align*}
v(t, 0) - v(t, 2\pi) &= v_x(t, 0) - v_x(t, 2\pi) = 0 & (t \in [0, 2\pi]), \\
v(0, x) - v(2\pi, x) &= v(t, 0) - v(t, 2\pi) = 0 & (x \in [0, 2\pi]).
\end{align*}
\]

If \( \lambda \in \mathbb{R} \) the doubly \( 2\pi \)-periodic problem for

\[
utt - uxx - \lambda u = h(t, x)
\]

has a unique weak solution for every \( h \in L^2(J) \) if and only if \( \lambda \notin \Sigma \), where

\[
\Sigma = \{ n^2 - m^2 : (m, n) \in \mathbb{Z} \times \mathbb{Z} \} = \{ \ldots, \lambda_{-2}, \lambda_{-1}, \lambda_0 = 0, \lambda_1, \lambda_2, \ldots \}
\]

and \( \mathbb{Z} \) denotes the integers.

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Existence results for (1.2) with any of the usual boundary conditions (e.g., doubly periodic, periodic-Dirichlet) usually require \( f \) to be monotone in \( s \). The monotonicity enables one to work around difficulties created by the infinite multiplicity of the eigenvalue \( \lambda_0 = 0 \).

It is known that, with our boundary conditions, (1.2) has a solution for each \( h \in H \) if \( f \) is monotone in \( s \) and is asymptotically between (and bounded away from) two successive eigenvalues (Mawhin [M.1]), or if \( f \) is monotone in \( s \) and “jumps” (asymptotically, going from \(-\infty\) to \(+\infty\)) from one eigenvalue to the next, or to the one below (provided neither is \( \lambda_0 = 0 \)) (Willem [Wi]).

These results have been recently unified and generalized to include nonuniformities in the avoidance of \( \Sigma \) (Mawhin and Ward [M-W.1, M-W.2]). Here we show that \( f \) may jump across arbitrarily many eigenvalues of finite multiplicity and (1.3) may still be solvable for all \( h \in H \).

2. Statement of results. The following is our main result.

**Theorem 1.** Let \( f: J \times \mathbb{R} \to \mathbb{R} \) satisfy the Carathéodory conditions and (1.1). Let \( h \in L^2(J) \) and suppose:

1. \( f(t,x,s) \) is monotone nondecreasing in \( s \) for each \( (t,x) \in J \).
2. There is a number \( \alpha > 0 \) and a function \( \beta \in H \) such that
   \[
   |f(t,x,s)| \leq f(t,x,s) + \alpha|s| + \beta(t,x)
   \]
   for all \( (t,x,s) \in J \times \mathbb{R} \).
3. There is a number \( \eta_0 > 0 \) such that \( 0 < \eta_0 \leq \lim_{|s| \to \infty} s^{-1}f(t,x,s) \), uniformly in \( (t,x) \in J \).

Then there is a number \( \alpha_0 > 0 \) such that a weak solution to the doubly 2\( \pi \)-periodic problem for (1.2) exists whenever \( \alpha < \alpha_0 \).

**Remark 1.** One may instead assume \( f(t,x,s) \) is nonincreasing in \( s \) and the existence of \( \eta_0 < 0 \) with \( 0 > \eta_0 \geq \lim_{|s| \to \infty} s^{-1}f(t,x,s) \). One may also replace \( f \) in (c2) by \(-f\).

As a corollary we have the following result on jumping nonlinearities. Consider the equation

\[
(2.1) \quad u_{tt} - u_{xx} + \alpha_- u^- - \alpha_+ u^+ - g(t,x,u) = h(t,x)
\]

where \( u^+ = \max(u,0) \), \( u^- = \max(-u,0) \), and \( u = u^+ - u^- \). Suppose \( h \in L^2(J) \) and \( g: J \times \mathbb{R} \to \mathbb{R} \) satisfies the Carathéodory conditions and (1.1).

**Corollary 1.** Let \( \alpha_- \) and \( \alpha_+ \) be positive numbers. Suppose \( f(t,x,s) := -\alpha_- s^- + \alpha_+ s^+ + g(t,x,s) \) is monotone nondecreasing in \( s \) and

\[
\lim_{|s| \to \infty} s^{-1}g(t,x,s) = 0,
\]

uniformly for \( (t,x) \in s \). Then there is a number \( \alpha_0 > 0 \) such that (2.1) has a weak doubly 2\( \pi \)-periodic solution provided \( 0 < \alpha_- < \alpha_0 \) (or \( 0 < \alpha_+ < \alpha_0 \)). (\( \alpha_0 \) does not depend on \( \alpha_+, g, \) or \( h \)).

**Remark 2.** A similar corollary is true with \( \alpha_- \), \( \alpha_+ \) negative and \( f(t,x,s) \) nonincreasing in \( s \).

**Remark 3.** We note that if \( \alpha_- < \alpha_+ \) then we may have \( |\alpha_- , \alpha_+| \cap \Sigma \neq \emptyset \). Indeed, the interval \( |\alpha_- , \alpha_+| \) may contain any finite number of positive eigenvalues.
REMARK 4. The corollary may be viewed as a surjectivity result concerning operators of the form $Lu + \alpha_- u^- - \alpha_+ u^+$, where $L$ is the D'Alembertian realized in $L^2(J)$ with our boundary conditions.

3. Abstract formulation and proofs. Let $H = L^2(J)$ with inner product $(u, v) = \int_J uv \, dt \, dx$ and corresponding norm $\| \cdot \|$. Each $u \in H$ has a representation as a Fourier series of the form

$$u = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} \alpha_{mn} u_{mn},$$

where $u_{mn} = e^{i(mt+nx)}$ for $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and $\alpha_{m,n} \in \mathbb{C}$ with $\alpha_{m,n} = \overline{\alpha_{-m,-n}}$ to make the sum real.

Let

$$D(L) = \left\{ u \in H : \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} (n^2 - m^2)^2 |\alpha_{mn}|^2 < \infty \right\}.$$

Define $L : D(L) \subseteq H \to H$ by

$$Lu = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} (n^2 - m^2) \alpha_{mn} u_{mn} \quad \text{for } u \in D(L).$$

Let $F : H \to H$ be the substitution operator defined by $F$. It is known that $u$ is a weak solution of the doubly $2\pi$-periodic problem for (1.2) if and only if $u \in D(L)$ and

$$(3.1) \quad Lu - F(u) = h$$

PROOF OF THEOREM 1. Let $\lambda_1$ be the first positive eigenvalue of $L$ (i.e., smallest positive number in $\Sigma$ so that $\lambda_1 = 1$). Choose $\epsilon_0 > 0$ so that $\epsilon_0 < \min(\lambda_1, \eta_0)$. For $\lambda \in ]0,1[$ consider the family of equations

$$(3.2) \quad Lu - (1 - \lambda) \epsilon_0 u - \lambda F(u) = \lambda h.$$  

The operator $(L - \epsilon_0 I)^{-1}$ is not compact. Nevertheless, it follows from a theorem in Willem's paper [Wi] (or see [M.2]) that it suffices to show that all possible solutions of (3.2) are bounded in $H$ independently of $\lambda \in ]0,1[$.

If $(u, \lambda)$ is a solution of (3.2) with $0 < \lambda < 1$ then

$$(3.3) \quad Lu - (1 - \lambda) \epsilon_0 u - \lambda f(t,x,u) = \lambda h(t,x),$$

and by taking inner products with $1$ we derive, since $(Lu,1) = 0$,

$$(3.4) \quad \lambda \int_J f(t,x,u) \, dt \, dx = -(1 - \lambda) \epsilon_0 \int_J u \, dt \, dx - \lambda \int_J h(t,x) \, dt \, dx.$$  

Taking absolute values in (3.3) and using (c2) we have a.e. on $J$:

$$|Lu(t,x)| \leq (1 - \lambda) \epsilon_0 |u(t,x)| + \lambda f(t,x,u(t,x)) + \alpha |u(t,x)| + \beta(t,x) + |h(t,x)|.$$  

Integrating over $J$ and using (3.4) we obtain

$$(3.5) \quad |Lu|_{L^1} \leq (2\epsilon_0 + \alpha) |u|_{L^1} + C_1$$

where $C_1$ is a constant.
For \( u \in H \) let us write \( u = u_0 + u_1 \) with \( u_0 \in \ker L \) and \( u_1 \in \ker L^\perp = \text{Range } L \).
It is known (cf., e.g., [L or C-H]) that there is a constant \( \mu > 0 \) such that for
\( u = u_0 + u_1 \in D(L), \)
\[ |u_1|_{L^\infty} \leq \mu |Lu_1|_{L^1}. \]

Thus for any solution \( u \) of (3.2) we have
\[ |u_1|_{L^\infty} \leq \mu |Lu_1|_{L^1} \leq \mu (2\varepsilon_0 + \alpha) |u|_{L^1} + \mu C_1 \]
and
\[ (3.6) \quad |u_1|_{L^\infty} \leq (2\varepsilon_0 + \alpha) C_2 |u| + C_3 \]
for some constants \( C_2 \) and \( C_3 \). Of course, by (3.5) we also have
\[ (3.7) \quad |Lu_1|_{L^1} \leq (2\varepsilon_0 + \alpha) C_4 |u| + C_1. \]

Taking the inner product of the expression on each side of (3.2) with \( u \) we derive
\[ (3.8) \quad (1 - \lambda) \varepsilon_0 |u|^2 + \lambda \int J f(t, x, u) u = (Lu, u) - \lambda (h, u). \]

By condition (c3) there is a number \( r > 0 \) such that
\[ f(t, x, s)s \geq \varepsilon_0 s^2 \]
for \( |s| \geq r \). Thus there is a function \( \gamma \in H \) with
\[ f(t, x, s)s \geq \varepsilon_0 s^2 - \gamma(t, x) \]
for all \((t, x, s) \in J \times \mathbb{R} \). From (3.8) we see that
\[ (1 - \lambda) \varepsilon_0 |u|^2 + \lambda \int J \varepsilon_0 |u|^2 dt dx - \int J \gamma dt dx \leq |(Lu, u)| + \|h\| \cdot \|u\|. \]

Thus there is a constant \( C \) with
\[ \varepsilon_0 |u|^2 \leq |(Lu_1, u_1)| + \|h\| \cdot |u| + C \]
\[ \leq |Lu_1|_{L^1} \cdot |u_1|_{L^\infty} + \|h\| \cdot |u| + C. \]

By (3.6) and (3.7) we now obtain
\[ \varepsilon_0 |u|^2 \leq [(2\varepsilon_0 + \alpha) C_4 |u| + C_1] [(2\varepsilon_0 + \alpha) C_2 |u| + C_3] + \|h\| \cdot |u| + C \]
and, hence,
\[ \varepsilon_0 |u|^2 \leq (2\varepsilon_0 + \alpha)^2 k_1 |u|^2 + k_2 |u| + k_3 \]
for some constants \( k_1, k_2, \) and \( k_3 \).

A subtraction yields
\[ (3.9) \quad [\varepsilon_0 - (2\varepsilon_0 + \alpha)^2 k_1] |u|^2 \leq k_2 |u| + k_3. \]

By now choosing \( \varepsilon_0 \) and \( \alpha_0 \) sufficiently small we can insure that, since \( \alpha < \alpha_0, \)
\[ \varepsilon_0 - (2\varepsilon_0 + \alpha)^2 k_1 > 0, \]
which, by (3.9), implies \( |u| < M \) for some constant \( M > 0 \). All possible solutions of (3.2) are thus bounded independently of \( \lambda \in ]0, 1[, \) and (3.1) has a solution.

**Proof of the Corollary.** We take
\[ f(t, x, s) = -\alpha_- s^- + \alpha_+ s^+ + g(t, x, s). \]
By hypothesis $f$ is monotone nondecreasing in $s$. Also
\[
|f(t, x, s)| \leq \alpha_- s^- + \alpha_+ s^+ + |g(t, x, s)|
\]
\[
\leq f(t, x, s) + 2\alpha_- s^- + 2|g(t, x, s)|.
\]
By hypothesis, for each $\varepsilon > 0$ there exists $\gamma_\varepsilon \in H$ with $|g(t, x, s)| \leq \varepsilon |s| + \gamma_\varepsilon (t, x)$. It follows that
\[
|f(t, x, s)| \leq f(t, x, s) + (2\alpha_- + 2\varepsilon) |s| + 2\gamma_\varepsilon (t, x),
\]
which shows (c2) holds. Since $\varepsilon > 0$ may be chosen arbitrarily small we can insure that $2\alpha_- + 2\varepsilon < \alpha_0$, where $\alpha_0$ is the number in Theorem 1, by requiring $\alpha_- < \alpha_0/2$ and then choosing $\varepsilon$. Since
\[
\lim_{|s| \to \infty} s^{-1} f(t, x, s) > \min(\alpha_-, \alpha_+) > 0,
\]
the corollary follows.

Remark 5. Instead of looking for solutions $2\pi$-periodic in $t$ and $x$ we could also formulate our results for solutions $\omega_1$-periodic in $t$ and $\omega_2$-periodic in $x$ if we insist that $\omega_1/\omega_2$ be a rational number. This would insure that the d’Alembertian with these boundary conditions is realized in $H$ by a selfadjoint operator having properties like those of $L$ above. If $\omega_1/\omega_2$ is irrational, small divisors appear in the right inverse of $L$ which lead to unsolved difficulties.

4. A counterexample. It is easy to see that the corollary is false if $\alpha_- = 0$ and $\alpha_+ > 0$. For example, if $g \equiv 0$ we have
\[
Lu - \alpha^+ u^+ = h,
\]
and by taking inner products with 1 we see that $h$ must satisfy $(h, 1) \leq 0$. Thus (4.1) cannot be solvable for all $h \in H$. In spite of this, (4.1) is certainly solvable for some $h \in H$. One might expect a solution if
\[
(h, 1) = \int_J h \, dt \, dx < 0.
\]
We show however that there may not be a solution even then.

It is easy to show that $u \in \ker L$ if and only if $u = p(t + x) + q(t - x)$ for some $p, q$ each $2\pi$-periodic on $\mathbb{R}$ with $p, q \in L^2(0, 2\pi)$.

Let $0 < \delta_1 < \delta_2 < \pi$ and $p : [0, 2\pi] \to \mathbb{R}$ be at least $C^2$ smooth and defined by
\[
p(s) = \begin{cases} 
3 & \text{if } |\pi - s| \leq \delta_1, \\
0 & \text{if } |\pi - s| \geq \delta_2 \text{ and } 0 \leq s \leq 2\pi, \\
0 \leq p(s) \leq 3 & \text{elsewhere}.
\end{cases}
\]
Extend $p$ $2\pi$-periodically to all of $\mathbb{R}$ and define $\phi \in H$ by
\[
\phi(t, x) = p(t + x) \quad \text{for } (t, x) \in J.
\]
Then $\phi \in \ker L$; indeed, $\phi$ is a smooth (classical) solution of $u_{tt} - u_{xx} = 0$ and $\phi$ is $2\pi$-periodic in each of $x$ and $t$.

By choosing $\delta_2$ sufficiently small we can insure that
\[
\int_J \phi \, dt \, dx < 4\pi^2.
\]
On the other hand, by choosing $\delta_1$ sufficiently close to $\delta_2$, one can insure that
\[ \int_{\Omega} \phi \, dt \, dx < \int_{\Omega} \phi^2 \, dt \, dx. \]

Now consider (4.1) with $h = \phi - 1$. We observe that
\[ \int_{\Omega} h \, dt \, dx = \int_{\Omega} \phi \, dt \, dx - 4\pi^2 < 0. \]

Suppose $u \in D(L)$ solves
\[ (4.2) \quad Lu - \alpha_u u^+ = \phi - 1. \]

Taking inner products with $\phi$ we obtain, since $(Lu, \phi) = 0$,
\[ 0 \geq -\alpha_u \int_{\Omega} u^+ \phi \, dt \, dx = \int_{\Omega} (\phi - 1) \phi \, dt \, dx > 0. \]

Thus (4.2) cannot have a solution.

If we now let $h = \mu(\phi - 1)$ with $\mu \gg 1$ we see that there still is not a solution even when $-\int_{\Omega} h$ is large.

REFERENCES


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