A NONCOMPLETELY CONTINUOUS OPERATOR ON $L_1(G)$
WHOSE RANDOM FOURIER TRANSFORM IS IN $c_0(\hat{G})$

N. GHOUSSOUB AND M. TALAGRAND

ABSTRACT. Let $T$ be a bounded linear operator from $L_1(G, \lambda)$ into $L_1(\Omega, \mathcal{F}, P)$, where $(G, \lambda)$ is a compact abelian metric group with its Haar measure, and $(\Omega, \mathcal{F}, P)$ is a probability space. Let $(\mu_\omega)$ be the random measure on $G$ associated to $T$; that is, $Tf(\omega) = \int_G f(t) \, d\mu_\omega(t)$ for each $f$ in $L_1(G)$.

We show that, unlike the ideals of representable and Kalton operators, there is no subideal $B$ of $M(G)$ such that $T$ is completely continuous if and only if $\mu_\omega \in B$ for almost $\omega$ in $\Omega$. We actually exhibit a noncompletely continuous operator $T$ such that $\mu_\omega \in l_{2+\varepsilon}(\hat{G})$ for each $\varepsilon > 0$.

I. Random measures associated to operators on $L_1$. Let $K$ be a separable compact Hausdorff space, $\lambda$ a Radon probability on $K$ and $(\Omega, \mathcal{F}, P)$ a probability space. Denote by $M(K)$ the space of all Radon measures on $K$. We will call random measure on $K$ every measurable map $\omega \mapsto \mu_\omega$ from $(\Omega, \mathcal{F}, P)$ into $M(K)$ when $M(K)$ is equipped with the $\sigma$-field generated by the $\sigma(M(K), C(K))$ topology.

The starting point of this paper is the following disintegration theorem established by Kalton [3] and Fakhoury [2].

(a) If $T$ is a bounded linear operator from $L_1(K, \lambda)$ into $L_1(\Omega, \mathcal{F}, P)$, then there exists an essentially unique random measure $(\mu_\omega)$ verifying the following properties:

(i) Each $f$ in $L_1(K, \lambda)$ belongs to $L_1(K, |\mu_\omega|)$ for $P$ almost all $\omega$.

(ii) $Tf(\omega) = \int_K f(t) \, d\mu_\omega(t)$ for $P$ almost all $\omega$.

(iii) $\int |\mu_\omega| \, dP(\omega) \leq M \cdot \lambda$, where $M = \|T\|$. 

(b) Conversely, every random measure $(\mu_\omega)$ verifying (i) and (iii) defines uniquely an operator $T$ from $L_1(K, \lambda)$ into $L_1(\Omega, \mathcal{F}, P)$ verifying (ii).

One can easily see that the lattice properties of the operators are compatible with the lattice properties of the random measures associated to them; that is, if $T = T^+ - T^-$ is the canonical decomposition of $T$ into its positive and negative parts, then $T^+$ and $T^-$ are given by the random measures $(\mu_\omega^+)$ and $(\mu_\omega^-)$ associated to $(\mu_\omega)$ by the Hahn-decomposition.

This representation allows us to associate to each order ideal $B$ in $M(G)$ an order ideal of operators $\mathcal{L}_B(L_1(K), L_1(\Omega))$ in $\mathcal{L}(L_1, L_1)$ via the formula

$$T \in \mathcal{L}_B \text{ iff } \mu_\omega \in B \text{ for almost all } \omega.$$

In this paper, we investigate whether a reasonable order ideal of operators $V$ in $\mathcal{L}(L_1, L_1)$ can be characterized in terms of its associated random measures belonging in a suitable order ideal in $M(K)$. A positive answer was given by Fakhoury [2] in the case of the space of representable operators (i.e. the operators $T$ such that there exists a Bochner integrable function $\varphi$: $K \to L_1(\Omega)$.
with \( T f = \int_{K} f \cdot \varphi(t) \, d\lambda(t) \) for all \( f \) in \( L_1(K) \). Fakhoury shows that \( T \) is representable if and only if \( \mu_\omega \) is absolutely continuous with respect to \( \lambda \) for almost all \( \omega \cdot (\mu_\omega \in M_{ac}(K)) \). Moreover, Kalton [3] proved that an operator \( T \) in \( \mathcal{L}(L_1(K), L_1(\Omega)) \) does not preserve a nontrivial projection band of \( L_1(K) \) (\( T \) is a non-Kalton operator) if and only if for almost all \( \omega \), \( \mu_\omega \) is a continuous measure on \( K \) (\( \mu_\omega \in M_c(K) \)).

Several subideals of \( M(K) \) can be defined if one assumes that \( K \) is a compact abelian group \( G \). For instance, let \( M_{ac}(G) \) be the space of strongly continuous measures on \( G \); these are the bounded regular Borel measures \( \mu \) on \( G \) such that \( |\mu|(g + H) = 0 \) for all \( g \) in \( G \), and all closed subgroups \( H \) of \( G \) such that \( G/H \) is infinite. Let \( M_0(G) \) be the ideal of measures whose Fourier transforms go to zero at infinity. Following Rosenthal [4], we also define \( M_\Lambda(G) \), the space of \( \Lambda \)-measures: these are the measures \( \mu \) such that for each \( \delta > 0 \), there exists \( \rho > 2 \) such that \( \{ \gamma; \gamma \in \hat{G}, |\mu(\gamma)| > \delta \} \) is a \( \Lambda_\rho \) set (i.e. the \( L_p \) and the \( L_1 \) norms are equivalent on the linear span of the characters in \( H \)). Note that

\[
M_{ac}(G) \subseteq M_0(G) \subseteq M_\Lambda(G) \subseteq M_{ac}(G) \subseteq M_c(G).
\]

One can define the corresponding order ideals of operators on \( L_1(G) \) via the above representation.

Recall that an operator \( T: L_1(G) \to L_1(\Omega) \) is said to be a completely continuous operator or a Dunford-Pettis operator (resp. an Enflo operator) if \( T(\text{Ball}(L_\infty(G))) \) is norm compact (resp. if there exists a subspace \( X \) of \( L_1(G) \), isomorphic to \( L_1 \), such that \( T \mid X \) is an isomorphism). It is well known [4] that a convolution operator \( T_\mu \) on \( L_1(G) \) is Dunford-Pettis if and only if \( \mu \in M_0(G) \) and \( T_\mu \) is a non-Enflo operator whenever \( \mu \in M_\Lambda(G) \). In the following, we shall prove that the random analogues to these properties do not hold, and that unlike the representable and the Kalton operators, there is no ideal in \( M(G) \) which can characterize the Dunford-Pettis and the non-Enflo operators.

We shall need the following fact [1]: If \( (X_n) \) is a sequence of mean zero Bernoulli random variables on a probability space \((\Omega, \mathcal{F}, P)\) then

\[
P \left( \left| \sum_{i=1}^{m} X_i \right| > \lambda \right) < e^{-2\lambda^2/m} \quad \text{for each } 0 \leq \lambda \leq m/2.
\]

**Lemma.** For each \( m \), there exists a nonnegligible set \( A_m \) in \( \{ -1, 1 \}^m \) such that \( \int_{A_m} \chi \, d\lambda \leq 2\sqrt{m} \cdot 2^{-m/2} \) for each character \( \chi \) of \( \{ 0, 1 \}^m \) which is not the identity.

**Proof.** Let \( \{ \chi_k; 1 \leq k \leq 2^m \} \) be an enumeration of the characters. For each \( k \), write \( \chi_k = \chi_{B_k^1} - \chi_{B_k^2} \), where \( B_k^2 \) is the complement of \( B_k^1 \), and let \( \{ x_j^i; 1 \leq j \leq 2^{m-1} \} \) be an enumeration for the elements of each set \( B_k^i \) (\( i = 1, 2 \)).

Define now, on the probability space of the subsets of \( \{ 0, 1 \}^m \) (which can be identified with \( \{ 0, 1 \}^{2m} \)), the following Bernoulli random variables:

\[
X_{x_j^i}(A) = \begin{cases} 
0 & \text{if } x_j \notin B_k^i \cap A, \\
1 & \text{if } x_j \in B_k^i \cap A,
\end{cases} \quad i = 1, 2.
\]

We get from (*) that

\[
P \left( A_i; \left| \sum_{j=1}^{2^{m-1}} X_{x_j^i}(A) - 2^{m-2} \right| > \alpha 2^{(m-1)/2} \right) < e^{-2\alpha^2}, \quad i = 1, 2,
\]
whenever $0 < \alpha \leq 2^{-3/2}$. In other words if we denote by $\Omega_{k}^i$ the set
\begin{equation*}
\{A; |\text{card}(A \cap B_k^i) - 2^{m-2}| \leq \alpha 2^{m-1/2}\},
\end{equation*}
we have
\begin{equation*}
P \left( \bigcap_{k=1}^{2m} \Omega_{k}^i \right) \geq 1 - \sum_{k=1}^{2m} P(\Omega_{k}^i) \geq 1 - 2^{m+1} \cdot e^{-2\alpha^2}.
\end{equation*}

If we take $\alpha = \sqrt{m}$, we get that there exists $A_m$, such that for each \{B_k^i; i = 1, 2, 1 \leq k \leq 2^m\},
\begin{equation*}
(\ast \ast) \quad |\text{card}(A_m \cap B_k^i) - 2^{m-2}| \leq \sqrt{m} \cdot 2^{m/2}.
\end{equation*}

Let $\chi_k = \chi_{B_k^1} - \chi_{B_k^2}$; we have
\begin{align*}
\left| \int_{A_m} \chi_k \, d\lambda \right| &= 2^{-m} \left| \text{card}(A_m \cap B_k^1) - \text{card}(A_m \cap B_k^2) \right| \\
&\leq 2 \cdot 2^{-m} \cdot \sqrt{m} \cdot 2^{m/2} = 2\sqrt{m} \cdot 2^{-m/2}.
\end{align*}

Now we can prove the following

**THEOREM.** There exists a compact abelian group $G$ and an Enflo operator $T: L_1(G) \to L_1(\{0, 1\}^N)$ such that the Fourier transform of the random measure associated to $T$ is in $l^{2+\varepsilon}$ for each $\varepsilon > 0$.

**PROOF.** For each $m$, define the random measure $\mu^m: \{0, 1\} \to \{-1, 1\}^m$ by $\mu_0^m = \lambda_{A_m}/\lambda(A_m)$ and $\mu_1^m = \lambda_{A_{m}}/\lambda(A_{m})$, where $\lambda_{B}$ denotes the restriction of the Haar measure on $\{-1, 1\}^m$ on the set $B$.

Let now $G$ be the group $\prod_m \{-1, 1\}^m$ and define the random measure $\mu: \{0, 1\}^N \to G$ by $\mu(x_1, x_2, \ldots) = \otimes_m \mu_{x(m)}^m$.

Let $T$ be the operator from $L_1(G) \to L_1(\{0, 1\}^N)$, defined by
\begin{equation*}
Tf(x_1, x_2, \ldots) = \int_G f(t) \, d\mu(x_1, x_2, \ldots)(t).
\end{equation*}

To prove that $T$ is bounded, it is enough to notice that for each set $A$ in $G$, we have
\begin{equation*}
\int \mu(x_1, x_2, \ldots)(A) \, d\lambda(x_1, x_2, \ldots) = \mu(A),
\end{equation*}
where $\lambda$ is the Haar measure on $\{0, 1\}^N$ and $\mu$ is the Haar measure on $G = \prod_m \{-1, 1\}^m$. For each finite set $F \subseteq N$, let $A_F = \{\chi; \chi = \prod_{k \in F} \chi_k \text{ and } \chi_k \text{ is a character on } \{-1, 1\}^k 	ext{ which is different from one}\}$. Note that
\begin{equation*}
\sum_{\chi \in A_F} |TX|^{2+\varepsilon} \leq 2^{L_F} \left( \prod_{k \in F} 2\sqrt{k2^{-k/2}} \right)^{2+\varepsilon} \leq \prod_{k \in F} (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2},
\end{equation*}
where $L_F = \sum_{k \in F} k$. 

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Therefore, for each \( q \),
\[ \sum_{\text{card}(F) = q} \left| T\chi \right|^{2+\varepsilon} \leq \sum_{\text{card}(F) = q} \prod_{k \in F} (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2} \leq (q!)^{-1} \left( \sum_{k} (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2} \right)^q. \]

Finally,
\[ \sum_{\chi} \left| T\chi \right|^{2+\varepsilon} \leq \exp \left( \sum_{k} (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2} \right) < \infty \text{ for each } \varepsilon > 0. \]

Hence for each \( x = (x_1, x_2, \ldots) \) in \( \{0, 1\}^N \), \( \hat{\mu}_x \in l_{2+\varepsilon}(\hat{G}) \).

Now let \( B_m = \{(y_n) \in \prod_n \{-1, 1\}^n; y_m \in A_m\} \). Note that
\[ TX_{B_m}(x) = \mu_{x_m}^m(A_m) = \begin{cases} 1 & \text{if } x_m = 0, \\ 0 & \text{if } x_m = 1, \end{cases} \]
from which follows that \( T \) is not a Dunford-Pettis operator. Moreover, if \( \mathcal{F} \) is the \( \sigma \)-field generated by \( \{B_m; m \in \mathbb{N}\} \) and if we let \( S \) be the operator \( T \) restricted to \( L_1[G, \mathcal{F}] \), one can easily see that the random measures \( \nu(x_1, x_2, \ldots) \) associated to \( S \) are Dirac measures, which implies that \( S \) is a Kalton operator and that \( T \) is an Enflo operator.

**COROLLARY.** There is no subset \( B \) and \( M(G) \) such that the ideal of Dunford-Pettis operators (or the non-Enflo operators) is equal to \( L_B(L_1(G)) \).

**PROOF.** If there exists such a \( B \) for the Dunford-Pettis operators (resp. non-Enflo operators) then by considering convolution operators we must have \( M_0(G) \subseteq B \) (resp. \( M_A(G) \subseteq B \)). But the above example shows the existence of an Enflo operator \( T \) such that the corresponding random measures belong to \( M_0(G) \) and hence to \( B \), which is a contradiction.

**REMARK.** There exists a Dunford-Pettis operator on \( L_1 \) of a group \( G \) such that none of the random measures associated to it belongs to \( M_0(G) \). For that, it is enough to take \( G = \prod_n \{-1, 1\}^{(0,1)^n} \) and, for each \( s \) in \( (0,1)^n \), let \( A_s^n = \{y \in \{-1, 1\}^{(0,1)^n}; y(s) = 1\} \) and \( \nu^n_s = 2\hat{\mu}_{A_s^n} \) (the restriction of the Haar measure \( \mu^n \) of \( \{-1, 1\}^{(0,1)^n} \) on the set \( A_s^n \)). We leave it to the reader to check that the operator \( T: L_1(G) \rightarrow L_1(\prod_n \{0, 1\}^n) \) associated to the random measure \( \nu: \prod_n \{0, 1\}^n \rightarrow M(G) \) defined by \( \nu(x_1, x_2, \ldots) = \bigotimes_n \nu^n_s \) verifies the claimed properties.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA V6T 1W5

DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ PARIS VI, JUSSIEU, FRANCE