ON SUMMABILITY OF FOURIER SERIES AT A POINT
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ABSTRACT. In this paper summability of Fourier series by a regular linear method of summation determined by a triangular matrix, has been studied and various results—some known and some new—on Cesàro and Nörlund summability have been deduced. A convergence criterion has also been obtained.

1. Let $C = (c_{n,k})$, $k = 0, 1, 2, \ldots, n$, be a triangular matrix and let

$$t_n = \sum_{k=0}^{n} c_{n,k} s_k,$$

where $\{s_k\}$ is a given sequence of numbers. If $t_n \to s$ as $n \to \infty$, $\{s_n\}$ is called summable $(C)$ to $s$. In this paper we assume $c_{n,k} \geq 0$ for $k = 0, 1, 2, \ldots, n$, and $\sum_{k=0}^{n} c_{n,k} = 1$. Then a necessary and sufficient condition for regularity of the method $(C)$ is

$$\lim_{n \to \infty} c_{n,k} = 0 \quad \text{for each } k.$$

In the case

$$c_{n,k} = \frac{A_{n-k}}{A_n}, \quad \alpha \geq 0,$$

where $\{A_n\}$ is determined by the identity

$$(1-x)^{-\alpha} = \sum_{0}^{\infty} A_n^{-1} x^n \quad (|x| < 1),$$

the method $(C)$ reduces to the well-known Cesàro method $(C, \alpha)$. For

$$c_{n,k} = p_{n-k}/P_n, \quad P_n = p_0 + p_1 + \cdots + p_n > 0,$$

the method $(C)$ reduces to the Nörlund method $(N, p)$. In the case $p_n = 1/(n+1)$, the Nörlund method $(N, 1/(n+1))$ is also known as the harmonic method.

Let $f$ be a Lebesgue integrable periodic function with period $2\pi$ and let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{0}^{\infty} A_n(x).$$

We write

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\},$$

$$\Phi(t) = \int_{0}^{t} |\phi(u)| \, du \quad \text{and} \quad s_n(x) = \sum_{0}^{n} A_k(x).$$

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Let
\[ C_n(k) = \sum_{m=0}^{k} c_{n,n-m} \]
and, for \( u \geq 0 \), define \( C_n(u) = C_n([u]) \), where \([u]\) is the greatest integer function.

Throughout the paper \( K \) is used to denote an absolute constant, not necessarily the same at each occurrence.

2. We establish the following

**THEOREM.** Let \( \{c_{n,k}\} \) be nondecreasing with respect to \( k \). Let \( \chi \) be a positive function defined over \((0, \infty)\) such that as \( n \to \infty \), (i) \( n\chi(n) = O(1) \) and (ii) \( \int_{1}^{n} \chi(u)C_n(u) \, du = O(1) \). Then if \( \Phi(t) = o(\chi(\pi/t)) \), as \( t \to 0^+ \), the series \( \sum A_n(x) \) is summable \((C)\) to \( f(x) \).

3. **Proof.** We have that \( \{c_{n,k}\} \) is nonnegative and nondecreasing in \( k \). Hence,
\[ (n-k)c_{n,k} \leq \sum_{m=k+1}^{n} c_{n,m} \leq 1. \]
Thus for each fixed \( k \), \( c_{n,k} \to 0 \) as \( n \to \infty \), that is, \((C)\) is a regular method.

In view of the fact that the convergence of Fourier series at a point is a local property of the generating function, we may take \( \phi(t) = 0 \) over \([\delta, \pi]\), where \( 0 < \delta < \pi \). We choose \( \delta \) such that \( \Phi(t) = o(\chi(\pi/t)) \) for \( t \in (0, \delta) \). Let
\[ t_n(x) = \sum_{k=0}^{n} c_{n,k}s_k(x). \]
Then we need to show that \( t_n(x) - f(x) = o(1) \) as \( n \to \infty \). After the Dirichlet integral, for \( n > \pi/\delta \),
\[ t_n(x) - f(x) = \sum_{k=0}^{n} c_{n,k}s_k(x) - f(x) = \frac{1}{\pi} \int_{0}^{\delta} \phi(t)L(n,t) \, dt \]
\[ = \frac{1}{\pi} \left\{ \int_{0}^{\pi/n} + \int_{\pi/n}^{\delta} \right\} = I_1 + I_2, \quad \text{say}, \]
where
\[ L(n,t) = \sum_{k=0}^{n} \frac{c_{n,k} \sin \left( k + \frac{1}{2} \right)t}{\sin \left( \frac{1}{2} t \right)}. \]
As
\[ |L(n,t)| \leq \pi \sum_{k=0}^{n} \left( k + \frac{1}{2} \right) c_{n,k} \leq \pi \left( n + \frac{1}{2} \right), \]
we get
\[ |I_1| \leq \left( n + \frac{1}{2} \right) \int_{0}^{\pi/n} |\phi(t)| \, dt = o(n\chi(n)) = o(1), \]
as \( n \to \infty \).

Next, in view of the order estimates of McFadden [4, Lemma 5.11],
\[ \left| \sum_{k=a}^{b} c_{n,n-k}e^{i(n-k)t} \right| \leq KC_n(\pi/t), \]
where $0 \leq a \leq b \leq \infty$, $0 < t \leq \pi$, and $n$ a positive integer, we obtain

\[ |I_2| \leq K \int_{\pi/n}^{\delta} \frac{|\phi(t)| C_n(\pi/t)}{t} \, dt \]

\[ = K \sum_{k=r}^{n-1} \int_{\pi/(k+1)}^{\pi/k} \frac{|\phi(t)| C_n(\pi/t)}{t} \, dt + K \int_{\pi/r}^{\delta} \frac{|\phi(t)| C_n(\pi/t)}{t} \, dt, \]

where $r$ is a positive integer such that $\pi/r \leq \delta < \pi/(r + 1)$. As

\[ \int_{\pi/(k+1)}^{\pi/k} \frac{|\phi(t)| C_n(\pi/t)}{t} \, dt = \left[ \frac{C_n(\pi/t)}{t} \Phi(t) \right]_{\pi/(k+1)}^{\pi/k} + \int_{\pi/(k+1)}^{\pi/k} \frac{\Phi(t) C_n(\pi/t)}{t^2} \, dt, \]

\[ |I_2| \leq o(C_n(r)) + o(n\chi(n)C_n(n)) + K \int_{\pi/n}^{\delta} \frac{\Phi(t) C_n(\pi/t)}{t^2} \, dt \]

\[ = o(1) + o \left( \int_1^n \chi(u) C_n(u) \, du \right) = o(1). \]

This completes the proof of the Theorem.

4. The four corollaries in this section follow as a result of our Theorem.

**Corollary 1 (Hardy [2]).** Let $\alpha > 0$. If $\Phi(t) = o(t)$, as $t \to 0+$, then $\sum A_n(x)$ is summable $(C, \alpha)$ to $f(x)$.

The case $\alpha = 1$ is the classical result of Lebesgue (see [10, Theorem III 3.9]).

**Proof.** Let $\chi(u) = \pi/u$ and $c_{n,k} = A_{n-k}^{\alpha-1}/A_n^{\alpha}$. Then $\chi(\pi/t) = t$ and

\[ C_n(u) = \sum_{m=0}^{[u]} c_{n,n-m} = \sum_{m=0}^{[u]} A_{n-k}^{\alpha-1}/A_n^{\alpha} = \frac{A_{[u]}^{\alpha}}{A_n^{\alpha}}. \]

Thus $n\chi(n) = \pi$ and

\[ \int_1^n \chi(u) C_n(u) \, du = O(n^{-\alpha}) \int_1^n u^{\alpha-1} \, du = O(1) \quad \text{as } n \to \infty. \]

Hence all the hypotheses of the Theorem are satisfied and the result follows.

**Corollary 2. (i) (Siddiqi [6]).** If $\Phi(t) = o(t/\log(2\pi/t))$, as $t \to 0+$, then $\sum A_n(x)$ is summable $(N, 1/(n + 1))$ to $f(x)$.

(ii) If $\Phi(t) = o(t/\{\log(3\pi/t) \log(3\pi/t)\})$, as $t \to 0+$, then $\sum A_n(x)$ is summable $(N, 1/\{n + 2\log(n + 2)\}$).

(iii) If $\Phi(t) = o(t/\{\log(k\pi/t) \log_2(k\pi/t) \cdots \log_q(k\pi/t)\})$, as $t \to 0+$, then $\sum A_n(x)$ is summable $(N, 1/\{(n + k) \log(n + k) \cdots \log_{q-1}(n + k)\}$, to $f(x)$, where $\log_r x = \log(\log_{r-1} x)$, for $r \geq 2$, and $k$ is such that $\log_q k > 0$.

**Proof.** To deduce this corollary, note that, in case (i) taking

\[ \chi(u) = \frac{\pi}{u \log 2u} \quad \text{and} \quad c_{n,k} = \frac{1/(n + 1 - k)}{\sum_{0}^{n} 1/(k + 1)}, \]
we obtain
\[ \chi(\pi/t) = t/\log(2\pi/t), \]
\[ n\chi(n) = \pi/\log 2n = o(1) \quad \text{as } n \to \infty, \]
\[ C_n(u) = \sum_{0}^{[u]} 1/(m + 1)/\sum_{0}^{n} 1/(k + 1), \]
and thus
\[ \int_{1}^{n} \chi(u)C_n(u) \, du = O\left(\frac{1}{\log n}\right) \int_{1}^{n} \frac{1}{u} \, du = O(1). \]

Thus the hypotheses of the Theorem are satisfied and the result follows.

The choice of \( \chi, c_{n,k}, C_n(u), \) etc., is similarly suggested in each of the cases (ii) and (iii), and the proof of the corollary is completed.

COROLLARY 3. Let \( \{p_n\} \) be a nonnegative, nonincreasing sequence and let \( p(1/t) = p([1/t]) \) and \( P(1/t) = P([1/t]) \).

(i) (SINGH [7]). If (a) \( \Phi(t) = o(t/\log(\pi/t)) \) as \( t \to 0+ \), and
(b) \( \sum_{1}^{n} (P_k/k \log(k + 1)) = O(P_n), \)
then \( \sum A_n(x) \) is summable \((N,p)\) to \( f(x) \).

(ii) (PATI [5]). If (c) \( \Phi(t) = o(t/P(1/t)) \) as \( t \to 0+ \), and
(d) \( \log n = O(P_n), \)
then \( \sum A_n(x) \) is summable \((N,p)\) to \( f(x) \).

(iii) (SINGH [8]). If (e) \( \Phi(t) = o(p(1/t)/P(1/t)) \), as \( t \to 0+ \), then \( \sum A_n(x) \) is summable \((N,p)\) to \( f(x) \).

REMARKS. In their theorems both Pati and Singh have assumed an extra hypothesis on \( \{P_n\} \): \( "P_n \to \infty, \text{ as } n \to \infty". \)

PROOF. Since \( \{p_n\} \) is nonnegative and nonincreasing,
\[ (n + 1)p_n \leq p_0 + p_1 + \cdots + p_n = P_n. \]
Therefore \( np_n/P_n = O(1), \) as \( n \to \infty. \) Taking \( c_{n,k} = p_{n-k}/P_n \) we obtain
\[ C_n(u) = P(u)/P_n. \]

Case (i). Take \( \chi(u) = 1, \) for \( u \in (0, 2) \) and \( \chi(u) = \pi/(u \log u) \) for \( u \in [2, \infty). \)
Then for \( t \in (0, 1/2), \)
\[ \chi(\pi/t) = t/\log(\pi/t), \]
and, for \( n \geq 2, \)
\[ n\chi(n) = \pi/\log n. \]

Thus
\[ n\chi(n) = o(1) \quad \text{as } n \to \infty. \]
Also

\[\int_{1}^{n} \chi(u)C_n(u) \, du = \frac{P_1}{P_n} + \frac{\pi}{P_n} \int_{2}^{n} \frac{P(u)}{u \log u} \, du = \frac{P_1}{P_n} + \frac{\pi}{P_n} \sum_{k=2}^{n-1} \frac{P_k}{u \log u} \, du \leq \frac{1}{P_n} \left\{ P_1 + \pi \sum_{k=2}^{n-1} \frac{P_k}{k \log k} \right\} \leq K \left( \frac{1}{P_n} \right) \sum_{k=1}^{n} \frac{P_k}{k \log(k+1)} = O(1) \quad \text{as } n \to \infty,\]

and the hypotheses of the Theorem are satisfied.

Case (ii). Take \( \chi(u) = 1/uP(u) \). Then

\[n\chi(n) = 1/P(n) = O(1), \quad \text{as } n \to \infty,\]

and

\[\int_{1}^{n} \chi(u)C_n(u) \, du = \frac{1}{P_n} \int_{1}^{n} \frac{1}{u} \, du = \frac{\log n}{P_n} = O(1).\]

Case (iii). Let \( \chi(u) = p(u)/P(u) \). Then

\[n\chi(n) = np_n/P_n = O(1),\]

as shown earlier, and also

\[\int_{1}^{n} \chi(u)C_n(u) \, du = \frac{1}{P_n} \int_{1}^{n} p(u) \, du = O(1).\]

Thus in each of these cases, the hypotheses of the Theorem are satisfied and the corollary follows.

**Corollary 4 (A Convergence Criterion).** Let \( \chi \) be a decreasing function such that \( \int_{1}^{n} \chi(u) \, du = O(1) \). If \( \Phi(t) = o(\chi(\pi/t)) \), as \( t \to 0+ \), then \( \sum A_n(x) \) converges to \( f(x) \).

In particular, if \( \chi(\pi/t) \) denotes any of the following:

(i) \( t/(\log(2\pi/t))^{1+\varepsilon} \),
(ii) \( \{\log(k\pi/t)(\log \log(k\pi/t))^{1+\varepsilon}\} \ldots \) where \( \varepsilon > 0 \) and \( k \) is appropriately chosen, then \( \Phi(t) = O(\chi(\pi/t)) \) implies that \( \sum A_n(x) \) converges to \( f(x) \).

**Remarks.** This result may be compared with the corresponding classical results on nonconvergence of a Fourier series at a point of continuity, e.g. see [10, Theorem VIII 2.4, p. 303]. Thus, in the suggested particular cases, \( \varepsilon > 0 \) may not be replaced by \( \varepsilon = 0 \). For other alternate convergence criteria involving the case \( \varepsilon = 0 \), see [3, Theorems 3, 10; 9, Theorems 2, 3].

We shall need the following result for a proof of Corollary 4.

**Lemma [1].** Let \( \{p_n\} \) satisfy the Kaluza conditions:

for \( n \geq 0 \), \( p_n > 0 \) and \( p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1.\)
Then if \( \{P_n\} \) is bounded, the method \((N,p)\) is ineffective, i.e. only convergent sequences are summable by the method.

**Proof of Corollary 4.** We first note that as \( \chi \) is decreasing,
\[
n\chi(n) \leq \int_1^n \chi(u) \, du = O(1).
\]
Now choosing \( c_{n,k} = P_{n-k}/P_n \) such that \( \{p_n\} \) satisfies the requirements of the Lemma (e.g. \( \{p_n\} \) may be taken to be one of the sequences
\[
\left\{ \frac{1}{(n+1)(n+2)} \right\}, \quad \left\{ \frac{1}{2^n} \right\}, \quad \left\{ \frac{1}{(n+2)(\log(n+2))^{1+\varepsilon}} \right\}, \quad \varepsilon > 0,
\]
ecc., we see that the hypotheses of the Theorem are satisfied, and thus we complete the proof.

In the case of the particular instances cited, we note that
\[
\Phi(t) = O(t/(\log(2\pi/t))^{1+\varepsilon}), \quad \text{as } t \to 0+
\]
implies that
\[
\Phi(t) = o(t/(\log(2/t))^{1+2/\varepsilon}), \quad \text{as } t \to 0+,\]
and similarly in the other cases, and then the results as claimed follow.

**References**


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