M-ACCRETIVE OPERATORS WITH M-DISPERSIVE RESOLVENTS

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Abstract. We characterize linear m-accretive operators with m-dispersive resolvents. T is linear and m-accretive, with \((X + T)^{-1}\) m-dispersive, if and only if the sequence \(\left\{ n!\phi(\lambda + T)^{n+1}x \right\}_{n=0}^{\infty} \) equals the moments of a positive measure on the positive real line, for sufficiently many \( \phi \in X^* \), \( x \in X \).

Introduction. Classical analysis is often very useful in linear operator theory. In another paper, we show that T is a spectral operator of scalar type when the sequence \(\left\{ \phi(T^n x) \right\}_{n=0}^{\infty} \) equals the moments of a positive measure, for sufficiently many \( \phi \in X^* \), \( x \in X \). In this paper, we consider the sequence \(\left\{ n!\phi(\lambda + T)^{n+1}x \right\}_{n=0}^{\infty} \), for some positive \( \lambda \), and again apply the classical analysis of the moment problem.

We say that a sequence of real numbers \(\{a_n\}_{n=0}^{\infty} \) satisfies a Stieltjes moment condition if \(\sum_{k=0}^{n} a_k \geq 0 \) and \(\sum_{k=0}^{n} a_k a_{k+1} > 0 \), for all finite sequences \(\{a_k\} \) of complex numbers. These conditions are satisfied if and only if there exists a positive measure \( \mu \) so that

\[ a_n = \int_{0}^{\infty} t^n d\mu(t) \quad \text{for all } n \quad \text{(see [3])}. \]

We will say \(\{a_n\}_{n=0}^{\infty} \) is Stieltjes if, in addition, \(\sum_{k=0}^{n} \inf a_k \leq n \) \(\frac{1}{2k} \) is infinite. When \(\{a_n\}_{n=0}^{\infty} \) is Stieltjes, the positive measure \( \mu \) is unique.

If T is a linear operator, then a vector \( x \) is quasi-analytic for \( T \) if \(\sum_{n=0}^{\infty} \inf a_k \leq n \) \(\|T^n x\|^{-1} \) is infinite. \(\mathscr{D}_q(T) \) is the set of all quasi-analytic vectors for \( T \) (see [1]).

T is m-accretive if it is densely defined and \( -T \) generates a 1-parameter contraction semigroup. If the semigroup is positive, on a Banach lattice, T is m-dispersive.

See [2] for other terminology in operator theory, such as accretive and core.

Theorem. Suppose \( X \) is a Banach lattice, \( T \) is closed, densely defined and accretive, and there exists \( \mathcal{D} \subseteq X^+ \cap \mathcal{D}_q(T) \) such that \( (\lambda + T)(\mathcal{D}) \subseteq \mathcal{D} \) and \( \text{span } (\mathcal{D}) \) is a core for \( (\lambda + T)^2 \), for some \( \lambda > 0 \). Then T is m-accretive, with \( (\lambda + T)^{-1} \) m-dispersive if and only if the sequence \(\left\{ n!\phi(\lambda + T)^{n+1}x \right\}_{n=0}^{\infty} \) satisfies a Stieltjes moment condition for all \( \phi \in (X^*)^+ \), \( x \in \mathcal{D} \).
Proof. Suppose $T$ is $m$-accretive, with $(\lambda + T)^{-1}$ $m$-dispersive.

We have, for all $\phi$ in $X^*$, $x$ in the domain of $T$, the resolvent formula

$$n!\phi(\lambda + T)^{n+1}x = \int_0^\infty s^n\phi(e^{-s(\lambda + T)^{-1}}x)\,ds.$$ 

When $\phi \in (X^*)^+$, $x \in \mathcal{D}$, then $e^{-s(\lambda + T)^{-1}}$, the semigroup generated by $(\lambda + T)^{-1}$ is positive, so $\phi(e^{-s(\lambda + T)^{-1}}x)\,ds$ is a positive measure. Thus, \( \left\langle n!\phi(\lambda + T)^{n+1}x \right\rangle_{n=0}^\infty \)

satisfies a Stieltjes moment condition. (Note that the density of the domain of $T$ was not needed here.)

Conversely, suppose \( \left\langle n!\phi(\lambda + T)^{n+1}x \right\rangle_{n=0}^\infty \) satisfies a Stieltjes moment condition, for all $\phi$ in $(X^*)^+$, $x$ in $\mathcal{D}$.

Since $x \in \mathcal{D}(T)$, the sequence is Stieltjes. Thus, if $\phi \in (X^*)^+$, $x \in \mathcal{D}$, there exists a unique positive measure $\mu_{\phi,x}$ such that

$$n!\phi(\lambda + T)^{n+1}x = \int_0^\infty t^n\,d\mu_{\phi,x}(t) \quad \text{for all } n.$$ 

Note that

$$\int_0^\infty t^n\int_0^\infty d\mu_{\phi,x}(s)\,dt = \int_0^\infty \int_0^\infty t^n\,d\mu_{\phi,x}(s) = \int_0^\infty s^{n+1}d\mu_{\phi,x}(s) = n!\phi(\lambda + T)^{n+2}x$$

Since the positive measure in (1) is unique, we have

$$\int_0^\infty d\mu_{\phi,x}(s)\,dt = d\mu_{\phi,(\lambda + T)x}(t), \quad \text{for all } \phi \in (X^*)^+, x \in \mathcal{D}.$$ 

This implies that $d\mu_{\phi,(\lambda + T)x}(t)/dt$ exists and is a continuous function of $t$. For each $t$, define $F(t): (\lambda + T)^2(\mathcal{D}) \to X^{**}$ by

$$[F(t)y](\phi) = \frac{d\mu_{\phi,y}(t)}{dt},$$ 

if $\phi$ is positive. For arbitrary $\phi \in X^*$, there exist positive $\phi_1$, $\phi_2$, $\phi_3$, $\phi_4$, with $\phi_1$ orthogonal to $\phi_2$, $\phi_3$ orthogonal to $\phi_4$ and $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4).$ Then

$$[F(t)y]\phi = [F(t)y]\phi_1 - [F(t)y]\phi_2 + i([F(t)y]\phi_3 - [F(t)y]\phi_4).$$ 

To see that $F(t)y \in X^{**}$, note that if $\phi \in (X^*)^+$,

$$|F(t)(\lambda + T)x(\phi)| = \int_0^\infty d\mu_{\phi,x}(s) \leq \int_0^\infty d\mu_{\phi,x}(s) = \phi(\lambda + T)x.$$ 

Since $F(t)(\lambda + T)x$ is positive, this implies that $\|F(t)(\lambda + T)x\| \leq \|\phi(\lambda + T)x\|$. Thus $\|F(t)\| \leq 1$, for all $t \geq 0$.

Note that $F(0)(\lambda + T)x = \int_0^\infty F(s)x\,ds = (\lambda + T)x$. Thus $F(0) = I$. Collecting all the facts about $F(t)$: For all $y \in (\lambda + T)^2(\mathcal{D})$, $\phi \in (X^*)^+$,

$$[F(t)y](\phi) \text{ is positive continuous function of } t,$$

$$F(0) = I.$$
To show that $T$ is $m$-accretive, we need to show that the range of $(\lambda + T)$ is dense. First, we will show that the range of $(\lambda + T)^3$ is dense in the range of $(\lambda + T)^2$, written $R(\lambda + T)^2$.

So suppose $\phi \in [R(\lambda + T)^3]^{\perp}$, the annihilator of $R(\lambda + T)^3$. There exist positive $\phi_1, \phi_2, \phi_3, \phi_4$ with $\phi_1$ orthogonal to $\phi_2$, $\phi_3$ orthogonal to $\phi_4$, so that $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$.

Suppose $y \in (\lambda + T)^2(\mathcal{D})$. Then, for all $n$, $0 = n!\phi(\lambda + T)^{n+1}y = \int_0^\infty t^n [F(t)y](\phi) \, dt$.

By (6), this implies that

\[
\int_0^\infty t^n [F(t)y](\phi_1) \, dt = \int_0^\infty t^n [F(t)y](\phi_2) \, dt.
\]

Since $[F(t)y](\phi_i)$ is positive, for $i = 1, 2$, the uniqueness of the positive measure in (1) implies that $[F(t)y]\phi_1 dt = [F(t)y]\phi_2 dt$. Since $F(t)$ is continuous, this implies that $[F(t)y]\phi_1 = [F(t)y]\phi_2$; in particular,

\[
\phi_1(y) = [F(0)y]\phi_1 = [F(0)y]\phi_2 = \phi_2(y).
\]

By an identical argument, $\phi_3(y) = \phi_4(y)$. Thus $\phi(y) = 0$.

Since the span of $\mathcal{D}$ is a core for $(\lambda + T)^2$, $(\lambda + T)^2(\mathcal{D})$ is total in $R(\lambda + T)^2$.

Since $\phi$ annihilates the total set $(\lambda + T)^2(\mathcal{D}), \phi \in [R(\lambda + T)^2]^{\perp}$.

Thus, since $T$ is closed, $R(\lambda + T)^2 = R(\lambda + T)^3$.

Since $T$ is accretive, this implies that $T$ is $m$-accretive, on $R(\lambda + T)^2$. This implies that $(\lambda + T)^{-1}$ is a bounded operator on $R(\lambda + T)^2$. Thus, if $x$ is in the domain of $T$, then $x = (\lambda + T)^{-2}(\lambda + T)^2x$ is in $R(\lambda + T)^2$. Since the domain of $T$ is dense, and $T$ is closed, $R(\lambda + T)^2$, and therefore $R(\lambda + T)$, equals $X$. Thus $T$ is $m$-accretive.

To show that $T$ has an $m$-dispersive resolvent, we will extend $F(t)$ to all of $\mathcal{D}$, and show that it equals the semigroup generated by $(\lambda + T)^{-1}$.

Differentiating both sides of (7) gives

\[
-F(t)y = F'(t)(\lambda + T)y \text{ for all } y \in (\lambda + T)^2(\mathcal{D}).
\]

This implies that $F(t)y = e^{-t(\lambda + T)^{-1}}y$ (the semigroup exists, because $(\lambda + T)^{-1}$ is bounded). Thus $F(t)(\lambda + T)^2x$ is a $C^\infty$ function of $t$, for all $x$ in $\mathcal{D}$.

By (2), this implies that $d\mu_{\phi,x}(t)/dt$ exists, and is a $C^\infty$ function of $t$, for all $x$ in $\mathcal{D}$, $\phi$ in $(X^*)^*$. Define $G(t): \mathcal{D} \to X^{**}$ exactly as $F(t)$ was defined in (3). Equations (4)–(8) all hold for $G(t)$.

Since $e^{-t(\lambda + T)^{-1}}x$ (equal to $G(t)x$) is positive, for all $x$ in $\mathcal{D}$, and $\mathcal{D}$ equals $X^+$, $e^{-t(\lambda + T)^{-1}}$ is positive, so that $(\lambda + T)^{-1}$ is $m$-dispersive, concluding the theorem.

The hypotheses are clearly satisfied when $T$ is bounded and accretive (letting $\mathcal{D} = X^+$). Note that $T$ is automatically $m$-accretive.
Corollary. Suppose X is a Banach lattice and T is bounded and accretive. Then \((\lambda + T)^{-1}\) is \(m\)-dispersive if and only if \(\left< n!\phi(\lambda + T)^{n+1}x \right>_{n=0}^{\infty}\) satisfies a Stieltjes moment condition for all \(\phi \in (X^*)^+, x \in X^+\).

References


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