M-ACCRETIVE OPERATORS WITH M-DISPERSE RESOLVENTS

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Abstract. We characterize linear m-accretive operators with m-dispersive resolvents. T is linear and m-accretive, with $(X + T)^{-1}$ m-dispersive, if and only if the sequence $\left\langle n! \phi(\lambda + T)^{n+1}x \right\rangle_{n=0}^{\infty}$ equals the moments of a positive measure on the positive real line, for sufficiently many $\phi$ in $X^*$, $x$ in $X$.

Introduction. Classical analysis is often very useful in linear operator theory. In another paper, we show that in the sequence $\left\langle \phi(T^n x) \right\rangle_{n=0}^{\infty}$ equals the moments of a positive measure, for sufficiently many $\phi$ in $X^*$, $x$ in $X$. In this paper, we consider the sequence $\left\langle n! \phi(\lambda + T)^{n+1}x \right\rangle_{n=0}^{\infty}$, for some positive $\lambda$, and again apply the classical analysis of the moment problem.

We say that a sequence of real numbers $(a_n)_{n=0}^{\infty}$ satisfies a Stieltjes moment condition if $\sum_{k=0}^{n} a_k a_{k+1} > 0$ and $\sum_{k=0}^{n} a_k a_{k+1} > 0$, for all finite sequences $(a_k)$ of complex numbers. These conditions are satisfied if and only if there exists a positive measure $\mu$ so that

$$a_n = \int_0^\infty t^n d\mu(t) \quad \text{for all } n$$

We will say $(a_n)_{n=0}^{\infty}$ is Stieltjes if, in addition, $\sum_{k=0}^{n} \inf_{k \leq n} |a_k|^{-1/2k}$ is infinite. When $(a_n)_{n=0}^{\infty}$ is Stieltjes, the positive measure $\mu$ is unique.

If $T$ is a linear operator, then a vector $x$ is quasi-analytic for $T$ if $\sum_{k=0}^{n} \inf_{k \leq n} \|T^k x\|^{-1/k}$ is infinite. $D_q(T)$ is the set of all quasi-analytic vectors for $T$ (see [1]).

$T$ is m-accretive if it is densely defined and $(-T)$ generates a 1-parameter contraction semigroup. If the semigroup is positive, on a Banach lattice, $T$ is m-dispersive.

See [2] for other terminology in operator theory, such as accretive and core.

Theorem. Suppose $X$ is a Banach lattice, $T$ is closed, densely defined and accretive, and there exists $D \subseteq X^+ \cap D_q(T)$ such that $(\lambda + T)(D) \subseteq D$ and span $(D)$ is a core for $(\lambda + T)^2$, for some $\lambda > 0$. Then $T$ is m-accretive, with $(\lambda + T)^{-1}$ m-dispersive if and only if the sequence $\left\langle n! \phi(\lambda + T)^{n+1}x \right\rangle_{n=0}^{\infty}$ satisfies a Stieltjes moment condition for all $\phi$ in $(X)^+$, $x$ in $D$.
PROOF. Suppose $T$ is $m$-accretive, with $(\lambda + T)^{-1}$ $m$-dispersive.

We have, for all $\phi$ in $X^*$, $x$ in the domain of $T$, the resolvent formula

$$n!\phi(\lambda + T)^{n+1} x = \int_0^\infty s^n \phi(e^{-s(\lambda + T)^{-1}} x) \, ds.$$ 

When $\phi \in (X^*)^+$, $x \in \mathcal{D}$, then $e^{-s(\lambda + T)^{-1}}$, the semigroup generated by $(\lambda + T)^{-1}$ is positive, so $\phi(e^{-s(\lambda + T)^{-1}} x) \, ds$ is a positive measure. Thus, $\left\langle n!\phi(\lambda + T)^{n+1} x \right\rangle_{n=0}^\infty$ satisfies a Stieltjes moment condition. (Note that the density of the domain of $T$ was not needed here.)

Conversely, suppose $\left\langle n!\phi(\lambda + T)^{n+1} x \right\rangle_{n=0}^\infty$ satisfies a Stieltjes moment condition, for all $\phi$ in $(X^*)^+$, $x \in \mathcal{D}$.

Since $x \in \mathcal{D}(T)$, the sequence is Stieltjes. Thus, if $\phi \in (X^*)^+$, $x \in \mathcal{D}$, there exists a unique positive measure $\mu_{\phi,x}$ such that

$$n!\phi(\lambda + T)^{n+1} x = \int_0^\infty t^n \, d\mu_{\phi,x}(t) \text{ for all } n.$$ 

Note that

$$\int_0^\infty t^n \int_t^\infty d\mu_{\phi,x}(s) \, dt = \int_0^\infty \int_0^s t^n \, d\mu_{\phi,x}(s) \, \frac{n!\phi(\lambda + T)^{n+1}}{n+1} x = \int_0^\infty t^n \, d\mu_{\phi,(\lambda - T)^{-1}} x.$$ 

Since the positive measure in (1) is unique, we have

$$\int_0^\infty d\mu_{\phi,x}(s) \, dt = d\mu_{\phi,(\lambda + T)^{-1}} x(t), \text{ for all } \phi \in (X^*)^+, x \in \mathcal{D}.$$ 

This implies that $d\mu_{\phi,(\lambda - T)^{-1}} x(t)/dt$ exists and is a continuous function of $t$. For each $t$, define $F(t): (\lambda + T)^2(\mathcal{D}) \to X^{**}$ by

$$[F(t)y](\phi) = \left. \frac{d\mu_{\phi,(\lambda - T)^{-1}} x(t)}{dt} \right|_{t=0}.$$

if $\phi$ is positive. For arbitrary $\phi \in X^*$, there exist positive $\phi_1, \phi_2, \phi_3, \phi_4$, with $\phi_1$ orthogonal to $\phi_2, \phi_3$ orthogonal to $\phi_4$ and $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$. Then

$$[F(t)y] \phi = [F(t)y] \phi_1 - [F(t)y] \phi_2 + i([F(t)y] \phi_3 - [F(t)y] \phi_4).$$

To see that $F(t)y \in X^{**}$, note that if $\phi \in (X^*)^+$, $\phi(\lambda + T)x$ is positive, this implies that $\|F(t)(\lambda + T)x\| \leq \|(\lambda + T)x\|$. Thus $\|F(t)y\| \leq 1$, for all $t \geq 0$.

Note that $F(0)(\lambda + T)x = \int_0^\infty F(s)x \, ds = (\lambda + T)x$. Thus $F(0) = I$. Collecting all the facts about $F(t)$: For all $y \in (\lambda + T)^2(\mathcal{D}), \phi \in (X^*)^+$,

$$[F(t)y](\phi) \text{ is positive continuous function of } t,$$

$$F(0) = I.$$
(6) \[ n!\phi(\lambda + T)^{n+1}y = \int_0^\infty t^n[F(t)y](\phi)\,dt, \]

(7) \[ \int_t^\infty [F(s)y](\phi)\,ds = [F(t)(\lambda + T)y](\phi). \]

To show that \( T \) is \( m \)-accretive, we need to show that the range of \((\lambda + T)^3\) is dense. First, we will show that the range of \((\lambda + T)^3\) is dense in the range of \((\lambda + T)^2\), written \( R(\lambda + T)^2 \).

So suppose \( \phi \in [R(\lambda + T)^2]^\perp \), the annihilator of \( R(\lambda + T)^3 \). There exist positive \( \phi_1, \phi_2, \phi_3, \phi_4 \) with \( \phi_1 \) orthogonal to \( \phi_2 \), \( \phi_3 \) orthogonal to \( \phi_4 \), so that \( \phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4) \).

Suppose \( y \in (\lambda + T)^2(\mathcal{D}) \). Then, for all \( n \), \( 0 = n!\phi(\lambda + T)^{n+1}y \). By (6), this implies that

\[ \int_0^\infty t^n[F(t)y](\phi_1)\,dt = \int_0^\infty t^n[F(t)y](\phi_2)\,dt. \]

Since \( [F(t)y](\phi_i) \) is positive, for \( i = 1, 2 \), the uniqueness of the positive measure in (1) implies that \( [F(t)y]\phi_1\,dt = [F(t)y]\phi_2\,dt \). Since \( F(t) \) is continuous, this implies that \( [F(t)y]\phi_1 = [F(t)y]\phi_2 \); in particular,

\[ \phi_1(y) = [F(0)y]\phi_1 = [F(0)y]\phi_2 = \phi_2(y). \]

By an identical argument, \( \phi_3(y) = \phi_4(y) \). Thus \( \phi(y) = 0 \).

Since the span of \( \mathcal{D} \) is a core for \((\lambda + T)^2\), \((\lambda + T)^3(\mathcal{D})\) is total in \( R(\lambda + T)^2 \).

Since \( \phi \) annihilates the total set \((\lambda + T)^2(\mathcal{D})\), \( \phi \in [R(\lambda + T)^2]^\perp \).

Thus, since \( T \) is closed, \( R(\lambda + T)^2 = R(\lambda + T)^3 \).

Since \( T \) is accretive, this implies that \( T \) is \( m \)-accretive, on \( R(\lambda + T)^2 \). This implies that \((\lambda + T)^{-1} \) is a bounded operator on \( R(\lambda + T)^2 \). Thus, if \( x \) is in the domain of \( T \), then \( x = (\lambda + T)^{-2}(\lambda + T)^2x \) is in \( R(\lambda + T)^2 \). Since the domain of \( T \) is dense, and \( T \) is closed, \( R(\lambda + T)^2 \), and therefore \( R(\lambda + T) \), equals \( X \). Thus \( T \) is \( m \)-accretive.

To show that \( T \) has an \( m \)-dispersive resolvent, we will extend \( F(t) \) to all of \( \mathcal{D} \), and show that it equals the semigroup generated by \((\lambda + T)^{-1}\).

Differentiating both sides of (7) gives

(8) \[ -F(t)y = F'(t)(\lambda + T)y \quad \text{for all} \quad y \in (\lambda + T)^2(\mathcal{D}). \]

This implies that \( F(t)y = e^{-t(\lambda + T)^{-1}}y \) (the semigroup exists, because \((\lambda + T)^{-1} \) is bounded). Thus \( F(t)(\lambda + T)^2x \) is a \( C^\infty \) function of \( t \), for all \( x \) in \( \mathcal{D} \).

By (2), this implies that \( d\mu_{\phi,x}(t)/dt \) exists, and is a \( C^\infty \) function of \( t \), for all \( x \) in \( \mathcal{D} \), \( \phi \) in \((X^*)^+\).

Define \( G(t) : \mathcal{D} \to X^{**} \) exactly as \( F(t) \) was defined in (3). Equations (4)–(8) all hold for \( G(t) \).

Since \( e^{-t(\lambda + T)^{-1}}x \) (equal to \( G(t)x \)) is positive, for all \( x \) in \( \mathcal{D} \), and \( \mathcal{D} \) equals \( X^+ \), \( e^{-t(\lambda + T)^{-1}} \) is positive, so that \((\lambda + T)^{-1} \) is \( m \)-dispersive, concluding the theorem.

The hypotheses are clearly satisfied when \( T \) is bounded and accretive (letting \( \mathcal{D} = X^+ \)). Note that \( T \) is automatically \( m \)-accretive.
Corollary. Suppose $X$ is a Banach lattice and $T$ is bounded and accretive. Then $(\lambda + T)^{-1}$ is $m$-dispersive if and only if $\left\langle n!\phi(\lambda + T)^{n+1}x\right\rangle_{n=0}^{\infty}$ satisfies a Stieltjes moment condition for all $\phi$ in $(X^*)^+$, $x$ in $X^+$.

References


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