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A RESONANCE PROBLEM IN WHICH THE NONLINEARITY MAY GROW LINEARLY

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ABSTRACT. The purpose of this paper is to study a semilinear two point boundary value problem of resonance type in which the nonlinear perturbation may grow linearly. A significant improvement of a recent result due to Cesari and Kannan is given.

We consider the boundary value problem

$$u'' + u + g(u) = h(x), \qquad u(0) = u(\pi) = 0, \tag{1}$$

where $h \in L^2[0,\pi]$ and g is continuous. If the numbers

$$\underline{g(\infty)} = \liminf_{\xi\to\infty} g(\xi) \quad \text{and} \quad \overline{g(-\infty)} = \limsup_{\xi\to-\infty} g(\xi)$$

are finite and

$$\overline{g(-\infty)} \int_0^\pi \sin x\,dx < \int_0^\pi h(x)\sin x\,dx < \underline{g(\infty)} \int_0^\pi \sin x\,dx, \tag{2}$$

then it follows from a slight variation of a well-known theorem due to Landesman and Lazer [**3**] that (1) has a solution. If the limits of g at $\pm\infty$ exist and $g(-\infty) < g < g(\infty)$, these conditions are also necessary for the solvability of (1). We show that if g satisfies an additional restriction, then (2) implies the existence of a solution of (1) even if one or both of the numbers $\underline{g(\infty)}$ and $\overline{g(-\infty)}$ is infinite. As will be shown in a remark after the proof, the following result is a strong improvement of the main result in [**1**].

For the case in which the inequalities (2) are reversed, see, for example, [**4**]. Other solvability conditions in the absence of the Landesman-Lazer conditions are given in [**5** and **6**].

THEOREM. *If there exist numbers γ and r, $r > 0$, with $0 < \gamma < 3$ such that*

$$g(\xi)/\xi \le \gamma, \quad |\xi| \ge r \tag{3}$$

and (2) *holds, then* (1) *is solvable.*

(By a solution we mean a function with absolutely continuous derivative which satisfies the boundary conditions and the differential equation a.e.)

PROOF. To prove the theorem we use the well-known continuation method of Leray and Schauder.

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For each s in $[0,1]$ we consider the boundary value problem

$$u'' + (1+\gamma)u + s(g(u) - \gamma u) = h(x), \qquad u(0) = u(\pi) = 0. \tag{4}$$

For v in $C[0,\pi]$, let $|v|_\infty = \max_{[0,\pi]} |v(t)|$. We claim that there exists a number R independent of s, $s \in [0,1]$, such that if u is a solution of (4), then $|u|_\infty < R$. In order to establish this claim, we first consider a certain decomposition of the function g. Let Φ be a smooth function such that $\Phi(\xi) = 0$ if $|\xi| \leq r$, $0 \leq \Phi(\xi) \leq 1$ for all ξ, and $\Phi(\xi) = 1$ if $|\xi| \geq 2r$. If we set $g_1(\xi) = \Phi(\xi)g(\xi)$ and $g_2(\xi) = (1 - \Phi(x))g(\xi)$, then $g_2(\xi)$ is bounded on $(-\infty,\infty)$, and (2) and (3) imply the existence of a number m such that

$$m \leq g_1(\xi)/\xi \leq \gamma \tag{5}$$

for $\xi \neq 0$. Setting $g(\xi)/\xi$ equal to 0 when $\xi = 0$, we may assume that (5) holds for all ξ, $\xi \in (-\infty,\infty)$.

If we assume that the claim is false, then there exists a sequence of numbers $\{s_n\}_{n=1}^\infty$ in $[0,1]$ and a corresponding sequence of functions $\{u_n\}_1^\infty$ such that u_n is a solution of (4) when $s = s_n$ and $|u_n|_\infty \geq n$ for all n. If we set $v_n = u_n/|u_n|_\infty$ for all n, then

$$v_n'' + v_n + p_n(x)v_n = h_n(x), \qquad v_n(0) = v_n(\pi) = 0, \tag{6}$$

where

$$p_n(x) = (1 - s_n)\gamma + s_n g_1(u_n(x))/u_n(x), \tag{7}$$

$$h_n(x) = [h(x) - s_n g_2(u_n(x))]/|u_n|_\infty. \tag{8}$$

From (5) we infer the existence of a number m_1 such that

$$m_1 \leq p_n(x) \leq \gamma \tag{9}$$

for all x. Moreover, since g_2 is bounded, it follows that $\lim_{n\to\infty} h_n = 0$ in $L^2[0,\pi]$. Since $|v_n|_\infty = 1$ for all n, it follows that the $L^2[0,\pi]$ norm of v_n'' is bounded independently of n. Since, by Rolle's theorem, v_n' vanishes somewhere on $(0,\pi)$, we see that the sequence $\{v_n'\}_1^\infty$ is equicontinuous and uniformly bounded on $[0,\pi]$. Therefore, the sequence $\{v_n\}_1^\infty$ is also equicontinuous and uniformly bounded on $[0,\pi]$. Hence, by Ascoli's lemma we may assume that $\lim_{n\to\infty} v_n(x) = w(x)$, and $\lim v_n'(x) = w'(x)$ uniformly on $[0,\pi]$, where $w \in C^1[0,\pi]$, $|w|_\infty = 1$, and $w(0) = w(\pi) = 0$. By (9), the sequence $\{p_n\}_1^\infty$ is bounded in $L^2[0,\pi]$. Hence we may assume that p_n converges weakly to a function $p(x)$ in $L^2[0,\pi]$.

It follows that $m_1 \leq p(x) \leq \gamma$ a.e. on $[0,\pi]$. This follows from (9), since, by Mazur's theorem, closed subsets of $L^2[0,\pi]$ are weakly closed.

From (6) we see that for x in $[0,\pi]$

$$v_n'(x) = v_n'(0) - \int_0^x (1 + p_n(t)v_n(t))\,dt - \int_0^x h_n(t)\,dt. \tag{10}$$

It follows, by letting $n \to \infty$ in (10), that

$$w'(x) = w'(0) - \int_0^x (1 + p(x))w(x)\,dx.$$

Therefore, w' is absolutely continuous and

$$w''(x) + (1 + p(x))w(x) = 0 \quad \text{a.e.}, \qquad w(0) = w(\pi) = 0.$$

We assert that $w(x) \neq 0$ for all x in $(0, \pi)$. Indeed, since $w(x) \not\equiv 0$ and $1 + p(x) \leq 1 + \gamma < 4$ a.e., if $w(\xi) = 0$ for some ξ in $(0, \pi)$, then, by the Sturm Comparison theorem, every nontrivial solution of $y'' + 4y = 0$ will have to vanish on each of the intervals $(0, \xi)$ and (ξ, π). Since $\sin 2t$ has one zero on $(0, \pi)$, the assertion follows. We will assume that $w(x) > 0$ for all x in $(0, \pi)$, and arrive at a contradiction. The alternative $w(x) < 0$ on $(0, \pi)$ will also lead to a contradiction.

Since w is a nontrivial solution of (11), $w'(0) > 0$ and $w'(\pi) < 0$. Since

$$v_n(x) = u_n(x)/|u_n|_\infty \to w(x)$$

as $n \to \infty$ in the norm of $C^1[0, \pi]$, it follows that $u_n(x) > 0$ on $[0, \pi]$ for large n. Consequently (2) implies that the function

$$(1 - s_n)\gamma u_n(x) + s_n g(u_n(x)) \equiv z_n(x)$$

is bounded below on $[0, \pi]$ independently of n. Since $|u_n|_\infty \to \infty$ as $n \to \infty$ and $w(x) > 0$ on $(0, \pi)$, it follows that $u_n(x) \to \infty$ uniformly on compact subintervals of $(0, \pi)$, and hence, by (2), we have

$$\int_0^\pi h(x) \sin x \, dx < \int_0^\pi \Big(\liminf_{n \to \infty} z_n(x)\Big) \sin x \, dx. \tag{12}$$

Multiplying the differential equation in (4) by $\sin x$ when $s = s_n$ and $u = u_n$ and integrating by parts, we obtain

$$\int_0^\pi z_n(x) \sin x \, dx = \int_0^\pi h(x) \sin x \, dx.$$

Since $z_n(x)$ is bounded below on $[0, \pi]$ independently of n, Fatou's lemma implies that

$$\int_0^\pi \Big(\liminf_{n \to \infty} z_n(x)\Big) \sin x \, dx \leq \lim_{n \to \infty} \int_0^\pi z_n(x) \sin x \, dx = \int_0^\pi h(x) \sin x \, dx,$$

which contradicts (12). Therefore, the existence of the a priori bound is established.

The proof now follows more or less along standard lines. Since the problem

$$Lu = u'' + u + \gamma u = 0, \qquad u(0) = u(\pi) = 0,$$

has no solution other than $u \equiv 0$, for any $f \in C[0, 1]$ the problem

$$Lu = f, \qquad u(0) = u(\pi) = 0,$$

has a unique solution, which we denote by $L^{-1}f$. By standard results, L^{-1} may be considered as a compact mapping from the Banach space $C[0, \pi]$ into itself. Let $G\colon C[0, \pi] \to C[0, \pi]$ be the Nemytskii map associated with g and let u_0 be the unique solution satisfying

$$Lu_0 = h, \qquad u_0(0) = u_0(\pi) = 0.$$

Since G is continuous and takes bounded subsets of $C[0, \pi]$ into bounded subsets of $C[0, \pi]$, the mapping $N\colon C[0, \pi] \times [0, 1] \to C[0, 1]$ given by

$$N(u, s) = u_0 + L^{-1}[s(\gamma u - G(u))]$$

is a compact homotopy. If $u = N(u, s)$ for some s in $[0, 1]$ then u is a solution of (4). Hence it follows from what has been shown above that there exists a number R,

$R > 0$, such that $|u|_\infty < R$. Let $D = \{u \in C[0,\pi] \,|\, |u|_\infty < R\}$. Since $u - N(u,s) \neq 0$ for all $(u,s) \in \partial D \times [0,1]$, it follows from the homotopy invariance theorem of degree theory (see, for example, [**2**]) that the Leray-Schauder degree $d(I - N(\cdot,s), D, 0)$ is constant for $s \in [0,1]$. Since $N(u,0) = u_0$ and our previous argument, applied to (4) with $s = 0$, shows that $|u_0| < R$, it follows that

$$1 = d(I - N(\cdot,0), D, 0) = d(I - N(\cdot,1), D, 0).$$

Hence $u = N(u,1)$ has a solution which is a solution of (1). This proves the theorem.

REMARK. Cesari and Kannan [**1**] consider problem (1) under the assumption that g is nondecreasing (actually, they write the differential equation in the form $u'' + u - g(u) = h$ and assume g to be nonincreasing). They assume that there exists a constant γ with $0 < \gamma < 0.443$ such that $|g(\xi)| < c + \gamma|\xi|$ for some c, $c \geq 0$. They further assume that

$$\limsup_{\xi\to\infty} g(\xi)/\xi = \gamma = -\liminf_{\xi\to-\infty} g(\xi)/\xi.$$

Obviously these assumptions imply that $\underline{g(\infty)} = \infty$ and $\overline{g(-\infty)} = -\infty$, so that (2) holds trivially. Also their assumptions imply (3) (with a different γ, $\gamma < 3$). Hence their result is a special case of our theorem. We emphasize that in Theorem 1 *it is not necessary* that g be *monotone*. Moreover, since the problem

$$u \quad + u + 3u = \sin 2t \qquad u(0) = u(\pi) = 0,$$

has no solution, the condition $\gamma < 3$ is sharp. Cesari and Kannon's condition that $\gamma < 0.443$ improves the condition $\gamma < 0.24347$ obtained earlier by Schechter, Shapiro and Snow in [**7**]. In [**7**] it was assumed that g is odd as well as nondecreasing.

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