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A SHORT PROOF OF THE CIMA-WOGEN $L(f)$ = CIRCLE THEOREM

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ABSTRACT. In [**1**] Cima and Wogen showed that if $f \in \operatorname{ball} \tilde{B}_0$ and $L(f)$ contains a circle γ, then $\gamma = L(f)$. This note presents a new and straightforward proof of Cima and Wogen's theorem.

Let $D = \{z\colon |z| < 1\}$ and let f be a holomorphic function in D. Following the notation of [**1**] let

$$M(f) = \sup\{|f'(z)|(1-|z|^2)\colon z \in D\}$$

and

$$B = \{f\colon f \text{ holomorphic in } D \text{ and } M(f) < \infty\},$$

B is the Bloch space with the norm $\|f\| = M(f) + |f(0)|$.

Now let

$$B_0 = \left\{ f \in B\colon \lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0 \right\},$$

$$\tilde{B} = \{f \in B\colon f(0) = 0\}, \qquad \tilde{B}_0 = B_0 \cap \tilde{B}$$

and

$$\operatorname{ball} \tilde{B}_0 = \{f \in \tilde{B}_0\colon M(f) \le 1\}.$$

If $R' = \{f\colon f \text{ holomorphic in } D \text{ and } M(f) \le 1\}$, then it is obvious that ball $\tilde{B}_0 \subset R'$.

For each $f \in R'$ let

$$L(f) = \{z \in D\colon |f'(z)|(1-|z|^2) = 1\}.$$

Now let $a \in D$ and let $r > 0$ such that $r < 1 - |a|$. The circle of center a and radius r is denoted by $C(a, r)$.

THEOREM 1. *For each $a \in D$ there is a sequence of real numbers $\{\delta_n(a)\}$ such that if $f \in R'$ and $L(f)$ contains a sequence of distinct points $\{z_k\}$ such that $|z_k| = r$ for every k, then $L(f) = C(a, r)$ and $r = \delta_n(a)$ for some n.*

PROOF. The function

$$h(z) = f''(z)/(zf''(z) + 2f'(z)) \tag{1}$$

is meromorphic in D and $h(z) = \bar{z}$ on $L(f)$ (see [**2**]).

Since $\{z_k\} \subset L(f)$ we have

$$h(z_k) = \bar{z}_k = r^2/z_k \quad \text{for every } k.$$

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Hence, by using the identity principle (note that $\{z_k\}$ has an accumulation point s with $|s| = r$) we get $h(z) = r^2/z$.

An integration on (1) yields $f'(z) = cz^\alpha$ with c a complex constant and $\alpha = 2r^2/(1-r^2)$. Condition $f \in R'$ implies $2r^2/(1-r^2) = n$ for some n. Consequently, $r = (n/(n+2))^{1/2} = \delta_n(0)$. Also we have that $c = \lambda/r^n(1-r^2)$ with $|\lambda| = 1$. Finally, it is clear that $|f'(z)|(1-|z|^2) < 1$ if $|z| \neq r$ since the real-valued function $g(x) = x^n(1-x^2)/r^n(1-r^2)$, $0 < x < 1$, satisfies $g(x) < 1$ if $x \neq r$. Then

$$f'(z) = \lambda z^n/r^n(1-r^2)$$

with $L(f) = C(0,r)$ and $r = \delta_n(0) = \delta_n$ (shortening).

Now let $a \neq 0$. By using a suitable bilinear transformation

$$\psi(z) = e^{i\theta}(z-p)/(1-z\overline{p}), \qquad |p| < 1,$$

and the first part of the proof, we obtain

$$f'(z) = \lambda \frac{[\psi(z)]^n}{\delta_n^n(1-\delta_n^2)}\psi'(z) \quad \text{for some } n,\ |\lambda| = 1.$$

A routine calculation gives

$$f'(z) = \lambda e^{i(n+1)\theta}\frac{(z-p)^n(1-|p^2|)}{\delta_n^n(1-\delta_n^2)(1-z\overline{p})^{n+2}}$$

and $L(f) = C(a,r)$.

From Theorem 1 and its proof we deduce

THEOREM 2 (CIMA AND WOGEN). *For each $a \in D$ there is a countable family of circles $\gamma_n(a)$ in D such that:*

(i) *If f is in the ball $\tilde{B}_0$ and $L(f)$ contains a circle γ, then $\gamma = \gamma_n(a)$ for some (a,n).*

(ii) *Conversely, given $\gamma_n(a)$ there is a function f_{na} in the ball $\tilde{B}_0$ so that $L(f_{na}) = \gamma_n(a)$.*

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