A SHORT PROOF OF
THE CIMA-WOGEN $L(f) = \text{CIRCLE THEOREM}$

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ABSTRACT. In [1] Cima and Wogen showed that if $f \in \text{ball } B_0$ and $L(f)$ contains a circle $\gamma$, then $\gamma = L(f)$. This note presents a new and straightforward proof of Cima and Wogen's theorem.

Let $D = \{z: |z| < 1\}$ and let $f$ be a holomorphic function in $D$. Following the notation of [1] let

$$M(f) = \sup\{|f'(z)|(1-|z|^2): z \in D\}$$

and

$$B = \{f: \text{ holomorphic in } D \text{ and } M(f) < \infty\},$$

$B$ is the Bloch space with the norm $\|f\| = M(f) + |f(0)|$.

Now let

$$\beta_0 = \{f \in \beta: \lim_{|z| \to 1} |f'(z)|(1-|z|^2) = 0\},$$

$$\tilde{B} = \{f \in B: f(0) = 0\}, \quad \tilde{B}_0 = B_0 \cap \tilde{B}$$

and

$$\text{ball } \tilde{B}_0 = \{f \in \tilde{B}_0: M(f) \leq 1\}.$$

If $R' = \{f: \text{ holomorphic in } D \text{ and } M(f) \leq 1\}$, then it is obvious that ball $\tilde{B}_0 \subset R'$.

For each $f \in R'$ let

$$L(f) = \{z \in D: |f'(z)|(1-|z|^2) = 1\}.$$

Now let $a \in D$ and let $r > 0$ such that $r < 1 - |a|$. The circle of center $a$ and radius $r$ is denoted by $C(a,r)$.

**Theorem 1.** For each $a \in D$ there is a sequence of real numbers $\{\delta_n(a)\}$ such that if $f \in R'$ and $L(f)$ contains a sequence of distinct points $\{z_k\}$ such that $|z_k| = r$ for every $k$, then $L(f) = C(a,r)$ and $r = \delta_n(a)$ for some $n$.

**Proof.** The function

$$h(z) = f''(z)/(zf''(z) + 2f'(z))$$

is meromorphic in $D$ and $h(z) = \bar{z}$ on $L(f)$ (see [2]).

Since $\{z_k\} \subset L(f)$ we have

$$h(z_k) = \bar{z}_k = r^2/z_k \quad \text{for every } k.$$
Hence, by using the identity principle (note that \( \{z_k\} \) has an accumulation point \( s \) with \(|s| = r\)) we get \( h(z) = \frac{r^2}{z} \).

An integration on (1) yields \( f'(z) = cz^\alpha \) with \( c \) a complex constant and \( \alpha = \frac{2r^2}{1 - r^2} \). Condition \( f \in R' \) implies \( 2r^2/(1 - r^2) = n \) for some \( n \). Consequently, \( r = \left(n/(n+2)\right)^{1/2} = \delta_n(0) \). Also we have that \( c = \lambda/r^n(1 - r^2) \) with \(|\lambda| = 1\).

Finally, it is clear that \( |f'(z)|(1 - |z|^2) < 1 \) if \(|z| \neq r\) since the real-valued function \( g(x) = x^n(1 - x^2)/r^n(1 - r^2) \), \( 0 < x < 1 \), satisfies \( g(x) < 1 \) if \( x \neq r \). Then

\[
f'(z) = \frac{\lambda z^n}{r^n(1 - r^2)}
\]

with \( L(f) = C(0, r) \) and \( r = \delta_n(0) = \delta_n \) (shortening).

Now let \( a \not= 0 \). By using a suitable bilinear transformation

\[
\psi(z) = e^{i\theta}(z - p)/(1 - zp), \quad |p| < 1,
\]

and the first part of the proof, we obtain

\[
f'(z) = \lambda \frac{[\psi(z)]^n}{\delta_n^n(1 - \delta_n^2)} \psi'(z)
\]

for some \( n \), \(|\lambda| = 1\).

A routine calculation gives

\[
f'(z) = \lambda e^{i(n+1)\theta} \frac{(z - p)^n(1 - |p|^2)}{\delta_n^n(1 - \delta_n^2)(1 - zp)^{n+2}}
\]

and \( L(f) = C(a, r) \).

From Theorem 1 and its proof we deduce

**THEOREM 2 (CIMA AND WOGEN).** For each \( a \in D \) there is a countable family of circles \( \gamma_n(a) \) in \( D \) such that:

(i) If \( f \) is in the ball \( B_0 \) and \( L(f) \) contains a circle \( \gamma \), then \( \gamma = \gamma_n(a) \) for some \((a, n)\).

(ii) Conversely, given \( \gamma_n(a) \) there is a function \( f_{na} \) in the ball \( B_0 \) so that \( L(f_{na}) = \gamma_n(a) \).

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**REFERENCES**
