A NOTE ON THE "HYPERBOLIC"
BOCHNER-RIESZ MEANS

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ABSTRACT. We consider the $L^p(\mathbb{R}^2)$ boundedness properties of the Fourier multiplier $m(\xi_1, \xi_2) = (1 - \xi_1^2 \xi_2^2)^+_{\alpha}$ for $\alpha > 0$. We prove that if $\alpha \geq \frac{1}{2}$, then $m$ is bounded on $L^p$, $1 < p < \infty$, and that if $\alpha > 0$, then $m$ is bounded on $L^p$, $\frac{4}{3} \leq p \leq 4$.

1. Introduction. In [6], El-Kohen defined the hyperbolic Bochner-Riesz multiplier of order $\lambda$ for $\mathbb{R}^2$ as

$$m^\lambda(\xi_1, \xi_2) = (1 - (\xi_1^2 \xi_2^2)^+)^\lambda.$$

If $(T^\lambda f)\hat{\cdot}(\xi) = m^\lambda(\xi) \hat{f}(\xi)$, he showed that $\|T^\lambda f\|_p \leq C_{p,\lambda} \|f\|_p$ for $1 < p < \infty$ provided that $\lambda > 1$ (a result which may be deduced from the Marcinkiewicz multiplier theorem; see Stein [8]). He also conjectured that the critical index $\lambda_0 = \inf\{\lambda > 0|\|T^\lambda f\|_p \leq C_{p,\lambda} \|f\|_p, 1 < p < \infty\}$ was indeed one. In this note we prove that, on the contrary, the hyperbolic Bochner-Riesz means behave like the usual spherical ones, so that $T^\lambda$ is bounded on $L^p$, $1 < p < \infty$, provided that $\lambda > \frac{1}{2}$, and that $T^\lambda$ is bounded on $L^p$, $\frac{4}{3} \leq p \leq 4$, provided that $\lambda > 0$.

THEOREM 1. If $1 < p < \infty$ and $\lambda > 0$, then

$$\|T^\lambda f\|_p \leq C_{p,\lambda} \|f\|_p$$

provided that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2}(\lambda + \frac{1}{2})$.

In proving Theorem 1 we may assume that supp\(\hat{f}\) is contained in the first quadrant, and to facilitate calculations we shall sometimes employ the rectangular hyperbolic coordinates $\alpha = \xi_1 \xi_2$ and $\beta = \xi_1^2 - \xi_2^2$ in Fourier transform space, so that $(T^\lambda f)\hat{\cdot}(\alpha, \beta) = (1 - \alpha^2)^+_{\lambda} \hat{f}(\alpha, \beta)$. We make the usual decomposition $(1 - \alpha^2)^+_{\lambda} = \sum_{k=0}^{\infty} 2^{-k\lambda} \phi_k(\alpha)$, with $\phi_k$ a smooth nonnegative bump function supported in $[1 - 2^{-k}, 1 - 2^{-k-2}]$. If $(T_k f)\hat{\cdot}(\alpha, \beta) = \phi_k(\alpha) \hat{f}(\alpha, \beta)$, we see that it suffices to obtain estimates for the form

$$\|T_k f\|_p \leq C_p (1 + k)^{\rho 2^{k(1 - 4/p)/2}} \|f\|_p$$

(for some constant $\rho > 0$) when $4 \leq p < \infty$. Now $\phi_0$ is easily seen to satisfy the hypotheses of the Marcinkiewicz multiplier theorem [8] and we turn to the estimates for $T_k$, $k \geq 1$. 

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2. Proof of Theorem 1. As outlined above, Theorem 1 will follow from

**Theorem 2.** Let \( \Phi \) be a fixed \( C^\infty \) function supported in \([-1, 1]\). For \( 0 < \delta < 1 \) and \( \alpha > 0 \), let \( \phi(\alpha) = \Phi((\alpha - 1)/\delta) \). Let \( (Tf)\gamma(\alpha, \beta) = \phi(\alpha)\hat{f}(\alpha, \beta) \). Then

\[
\|Tf\|_p \leq C_p \delta^{-1/2} \|f\|_p, \quad 1 < p < \infty,
\]

and

\[
\|Tf\|_4 \leq C \left( \log \frac{1}{\delta} \right)^{\mu} \|f\|_4,
\]

where \( C_p \) and \( C \) depend only on \( \max_{0 \leq j \leq 3} \|\Phi(j)\|_\infty \).

The proof of Theorem 2 follows the argument given by Córdoba in [3] to prove the Carleson and Sjölin theorem [1] concerning the spherical Bochner-Riesz multipliers. We shall assume that \( \delta^{-1/2} \in \mathbb{N}; \) for \( -\infty < k < \infty \) and \( 0 \leq j < \delta^{-1/2} - 1 \) let \( S^k_j = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : 2^k(1 + \delta^{-1/2}j) \leq \xi_1 \leq 2^k(1 + \delta^{-1/2}(j + 1)) \} \), let \( \{\gamma^k_j\} \) be a smooth partition of unity subordinate to \( \{S^k_j\} \), and let \( \tilde{\gamma}^k_j \) be a smooth function such that \( \tilde{\gamma}^k_j \equiv 1 \) on \( S^k_j \), and vanishes outside the two immediate neighbours of \( S^k_j \). (We choose \( \gamma^k_j \) and \( \tilde{\gamma}^k_j \) to depend only on \( \xi_1 \).) Let \( (A^k_j f)\gamma = \gamma^k_j \hat{f} \) and \( (B^k_j f)\gamma = \tilde{\gamma}^k_j \hat{f} \). Theorem 2 will follow from the following three propositions and a simple interpolation argument.

**Proposition 1.**

(a) \( \|Tf\|_p \leq C_p \delta^{-1/4} \|\sum_{k,j} |A^k_j Tf|^2\|_p^{1/2}, \quad 2 \leq p < \infty \).

(b) \( \|Tf\|_4 \leq C(\log \frac{1}{\delta})^{\mu} \|\sum_{k,j} |A^k_j Tf|^2\|_4^{1/2} \).

**Proposition 2.**

(a) \( \|(\sum_{k,j} |TA^k_j f|^2)^{1/2}\|_p \leq C\Phi C_p \delta^{-1/4} \|(\sum_{k,j} |B^k_j f|^2)^{1/2}\|_p \), \( 2 \leq p < \infty \).

(b) \( \|(\sum_{k,j} |TA^k_j f|^2)^{1/2}\|_4 \leq C\Phi (\log \frac{1}{\delta})^{\mu} \|(\sum_{k,j} |B^k_j f|^2)^{1/2}\|_4 \).

**Proposition 3.** \( \|(\sum_{k,j} |B^k_j f|^2)^{1/2}\|_p \leq C_p \|f\|_p, \quad 2 \leq p < \infty \).

Proposition 3 follows from Córdoba’s scissors lemma [4], and Propositions 1 and 2 follow from simple geometric observations about \( \{S^k_j \cap \text{supp} \phi\} \) together with an application of maximal theorems due to Nagel, Stein and Wainger and to Córdoba. (We shall assume without loss of generality that \( k \geq 0 \) when proving these propositions.)

**Proof of Proposition 1.**

(a) By the theory of vector-valued singular integrals [8]

\[
\|Tf\|_p \leq C_p \left\| \left( \sum_{k,j=0}^{\delta^{-1/2}-1} A^k_j Tf \right)^2 \right\|_p^{1/2} \leq C_p \delta^{-1/4} \left\| \left( \sum_{k,j} |A^k_j Tf|^2 \right)^{1/2} \right\|_p,
\]

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\[ \int |Tf|^4 = \int \overline{Tf} \cdot T\overline{f} = \int \left| \sum_{k,j} \sum_{k',j'} \gamma_{k,j}^* \phi * \gamma_{k',j'}^* \phi f \right|^2 \leq C \int \sum_{k,j} \sum_{k',j'} |\gamma_{k,j}^* \phi * \gamma_{k',j'}^* \phi f|^2 \]

\[ = C \int \sum_{k,j} \sum_{k',j'} |A_{k,j}^* Tf \cdot A_{k',j'}^* Tf|^2 = C \int \left( \sum_{k,j} |A_{k,j}^* Tf|^2 \right)^2. \]

Inequality (1) holds because of the following

**LEMMA 1.** No point of \( \mathbb{R}^2 \) belongs to more than \( C \) of the sets \( \{ (S_k^j \cap \text{supp } \phi) + (S_k^j \cap \text{supp } \phi) \} \).

**PROOF.** The lemma is true because of two properties: the global property that no point in \( \mathbb{R}^2 \) can be written in more than two ways as a sum of points in the first quadrant and lying on the rectangular hyperbola \( \xi_1 \xi_2 = 1 \) and the local property that if \( y = (\xi_1^1, \xi_2^1) + (\mu_1^1, \mu_2^1) \) with \( |\xi_1^1 \xi_2^2 - 1| < \delta \) and \( |\mu_1^1 \mu_2^2 - 1| < \delta, \xi_1^2 \leq \mu_2^1, j = (1, 2) \), then \( |1 - \xi_1^1/\xi_2^1| \leq c\delta^{1/2} \). To see this latter fact observe that if we perturb the solutions of the equation \( y = (\xi_1, \xi_2) + (\mu_1, \mu_2), \xi_1 \xi_2 = 1 = \mu_1 \mu_2 \) by allowing \( |\xi_1 \xi_2 - 1| \) and \( |\mu_1 \mu_2 - 1| < \delta, \) then \( \xi_1 \) and \( \mu_1 \) will change by a factor no larger than \( \delta(\xi_1 + \mu_1)/|\xi_1 \mu_2 - \xi_2 \mu_1| \). Thus the “marginal” change in \( \xi_1 \) and \( \mu_1 \) is less than \( \delta^{1/2} \) provided that \( |1 + \lambda|/|\lambda - 1/\lambda| \leq \delta^{-1/2} \), where \( \lambda = \xi_1/\mu_1 \) or \( \mu_1/\xi_1 \). If \( |1 + \lambda|/|\lambda - 1/\lambda| \geq \delta^{-1/2} \), however, then automatically \( |1 - \lambda| \leq c\delta^{1/2} \). □

Let \( M \) be the maximal function corresponding to averages over rectangles having a side parallel of one of the chords \( P_{kj} \) which joins the two points of \( \partial S_k^j \cap \{ \xi_1 \xi_2 = 1 \} \). Concerning \( M \) we have the following known results.

**LEMMA 2.** (a) (Córdoba [2]) \( ||M\omega||_2 \leq C(\log \frac{1}{\delta})^\rho ||\omega||_2 \).
(b) (Nagel, Stein and Wainger [7]) \( ||M\omega||_q \leq C_q \delta^{-1/2} ||\omega||_q, 1 < q < \infty \). □

To prove Proposition 2 we also need

**LEMMA 3.** The kernel \( L_j^k \) of the multiplier operator \( T A_{kj}^k \) satisfies
\[ (i) \int |L_j^k| \leq C, \]
\[ (ii) |L_j^k| * g(x) \leq CMg(x). \]

Here, \( C \) depends only on \( \max_{0 \leq j \leq 3} ||\Phi(j)||_\infty \).

**PROOF.** We need only see that \( \phi \gamma_j^k \) is a smooth bump function supported in a rectangle with width \( 10\delta/2^k \) and length \( 10\delta^{1/2}/2^k \) with longer side parallel to \( P_{kj} \). This is clear since the distance between \( P_{kj} \) and the hyperbola \( \xi_1 \xi_2 = 1 \) never exceeds \( 3\delta/2^k \). We now apply the integration by parts procedure as in [3] to obtain our result. □
PROOF OF PROPOSITION 2. For $p \geq 2$, let $\|\omega\|_{(p/2)'} \leq 1$. Then, following [3],

$$\int \left| \sum_{k,j} |TA^k_j f|^2 \omega \right| \leq C \int \left| \sum_{k,j} |L^k_j \ast B^k_j f|^2 \omega \right| dx$$

$$= C \int \left| \sum_{k,j} |B^k_j f|^2 (x) \ast \omega (x) \right| dx \leq C \int \sum_{k,j} |B^k_j f|^2 M \omega .$$

Lemma 2 now completes the proof of this proposition. $\Box$

REFERENCES


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