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DISTRIBUTION OF ALTERNATION POINTS IN UNIFORM POLYNOMIAL APPROXIMATION

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ABSTRACT. For a continuous function f on $[0,1]$, we discuss the points where the polynomial $P_n(x)$ of best uniform approximation deviates most from $f(x)$, and the signs of the difference $f(x) - P_n(x)$ alternate. We show that these points can be very irregularly distributed in $[0,1]$, even if f is entire.

1. Introduction. Problems discussed here concern uniform approximation of continuous functions by algebraic polynomials on [0, 1] and by trigonometric polynomials on the circle T.

By $Q_n := Q_n(f)$ (also written $Q_n(x)$), we always denote the polynomial of degree $\leq n$ of best uniform approximation to $f \in C[0,1]$. By Chebyshev's theorem, if f is not a polynomial, the Q_n are characterized by the existence of a sequence of at least $n+2$ "alternation points" x_k,

$$0 \leq x_1 < \cdots < x_N \leq 1, \tag{1.1}$$

where $\|f - Q_n\|_\infty = |f(x) - Q_n(x)|$, and where the signs of $f(x) - Q_n(x)$ alternate:

$$f(x_k) - Q_n(x_k) = (-1)^k \varepsilon \|f - Q_n\|, \quad \text{with } \varepsilon = \pm 1.$$

We denote by $N = N(n)$ the maximal length of an alternation sequence; then $N = n + 2 + s$, $s \geq 0$. This is equivalent to $Q_n = Q_{n+1} = \cdots = Q_{n+s} \neq Q_{n+s+1}$. By $A_n(f)$ we denote any set of alternation points (1.1) of f of maximal length $N(n)$.

How are the alternation points (1.1) distributed in [0, 1]? By a theorem of Kadec [**4**], for each f there are *infinitely many* n for which they are *equidistributed*. Let $d\mu = [x(1-x)]^{-1/2}\,dx$ be the Chebyshev measure on [0, 1]. Reformulating Kadec's theorem, and slightly improving his estimate, we have

THEOREM 1. *If $f \in C[0,\ 1]$ is not a polynomial, then there exists an infinite increasing sequence of positive integers $\{n_j\}$ so that with $N_j = N(n_j)$, for each interval $I \subset [0,\ 1]$,*

$$|A_{n_j}(f) \cap I| = \frac{\mu I}{\pi}(N_j + R_j), \quad |R_j| \leq C N_j^{1/2} \log N_j, \qquad j = 1, 2, \ldots, \tag{1.2}$$

where C is an absolute constant.

(Kadec has $N_j^{(1/2)+\varepsilon}$ instead of $N_j^{1/2} \log N_j$.)

It is natural to ask whether equidistribution takes place *for all large* n (instead of *some* large n), and if this is not true in general, whether it can be established at least when f is nice, say analytic. The following theorem is a counterexample.

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THEOREM 2. *There exists an entire function f and three infinite sequences of increasing positive integers $\{n_j^{(i)}\}$, $i = 1, 2, 3$, so that for each $\varepsilon > 0$, and all sufficiently large j,*

$$A_{n_j^{(1)}}(f) \subset [0, \varepsilon]; \qquad A_{n_j^{(2)}}(f) \subset [1-\varepsilon, 1],$$

while $A_{n_j^{(3)}}$ is equidistributed in $[0,\ 1]$ in the sense of (1.2).

This theorem holds also for trigonometric approximation—this is seen by the standard subtitution $x = \cos t$.

We need some lemmas.

LEMMA 1 (FREUD; see [**2**, p. 82]). *For each $f \in C[0,\ 1]$, there is a constant $M = M(f) > 0$ so that*

$$\|Q(g) - Q(f)\| \leq M\|g - f\|, \qquad g \in C[0, 1]. \tag{1.3}$$

A polynomial P_n of degree $\leq n$ is *θ-incomplete*, $0 < \theta < 1$, if it has the form $P_n(x) = \sum_{[n\theta]}^{n} a_k x^k$. The following is a variant of a theorem of the author; see [**5**].

LEMMA 2. *Let $0 < \sigma < \theta^2$. Then for each $\varepsilon > 0$ and each sufficiently large n, a θ-incomplete polynomial P_n with the property $|P_n(x)| \leq 1$, $x \in [\theta^2, 1]$ satisfies $|P_n(x)| < \varepsilon$ on $[0, \sigma]$.*

Incomplete polynomials with special properties have been studied in detail by Saff and Varga [**6**]. They proved

LEMMA 3. *For each $0 < \theta < 1$ and each n, there exists a θ-incomplete polynomial S_n with $\|S_n\| = 1$ which attains $\geq n(1-\theta)$ maxima and minima on $[\theta^2, 1]$, with alternating values ± 1.*

(The existence of such S_n follows also from considerations of Bernstein [**1**, p. 28]).

2. Proof of the theorem. Let $\theta_k \to 1$, $0 < \theta_k < 1$. The coefficients $a_j > 0$ will satisfy $\sum_{j=k}^{\infty} a_j \leq 2a_k$ and, in addition, $a_{j+1} = o(a_j)$, $j \to \infty$. We shall introduce another restriction for the a_j later.

The integers $n_k \geq 1$ will be defined by induction and subject to the inequality

$$n_k \leq (1-\theta_k)n_{k+1} - 2. \tag{2.1}$$

If the $n_1, \ldots, n_k$ are known, we select n_{k+1} so that (2.1) is satisfied, and, further, so that Lemma 2 applied to $S_{n_{k+1}}$ of Lemma 3 yields

$$|S_{n_{k+1}}(x)| < k^{-1} \quad \text{on } [0, \theta_k^3]. \tag{2.2}$$

Then we define

$$f(x) = \sum_{j=1}^{\infty} a_j \Pi_j, \tag{2.3}$$

where $\Pi_j(x) = S_{n_j}(x)$ for odd j, $= S_{n_j}(1-x)$ for even j. We let $f_k(x) = \sum_{j=1}^{k+1} a_j \Pi_j$, and compare $Q_n = Q_n(f)$ and $Q_n^* := Q_n(f_k)$, assuming that

$$n_k \leq n < (1-\theta_k)n_{k+1} - 2. \tag{2.4}$$

Let, for example, k be even. We have $Q_n^* = \sum_{j=1}^k a_j \Pi_j$, since the sum is a polynomial of degree $\leq n$, and since $f_k - Q_n^* = a_{k+1}\Pi_{k+1}$ has in $[\theta_k^2, 1]$ an alternating sequence of length $\geq n+2$. We also see that for such n, $\|f_k - Q_n^*\| = |a_{k+1}|$. We further have

$$\|f - f_k\| \leq \sum_{k+2}^{\infty} a_j \leq 2a_{k+2};$$

hence by Lemma 1,

$$\|Q_n - Q_n^*\| \leq 2Ma_{k+2}.$$

We write

$$f - Q_n = (f - f_k) + (f_k - Q_n^*) + (Q_n^* - Q_n) = a_{k+1}\Pi_{k+1} + h,$$

and have $\|h\| \leq 2(M+1)a_{k+2}$. This shows that $\|f - Q_n\| = a_{k+1}(1 + o(1))$. On the other hand, on $[0, \theta_k^3]$ we can use (2.2) and get

$$|f(x) - Q_n(x)| \leq 2(M+1)a_{k+2} + k^{-1}a_{k+1} = o(a_{k+1}).$$

Thus, all the alternation points of f, where $|f(x) - Q_n(x)| = \|f - Q_n\|$, lie in $[\theta_k^3, 1]$. And for odd k, they are contained in $[0, (1-\theta_k)^3]$.

To assure that f is entire, we have to take very small a_j. If M_j is the maximum of $|S_{n_j}(z)|$ on the disk $|z| \leq j$, it is sufficient to take $\sum_1^\infty a_j M_j < +\infty$. Finally, the third sequence $n_j^{(3)}$ exists by Theorem 1. □

3. Remarks. In spite of our main theorem one could still ask under what conditions one has *equidistribution*

$$|A_n(f) \cap I| = (\mu I/\pi)(N + o(1)) \tag{3.1}$$

for all $n \to \infty$. We can show that a sufficient condition is a modest but steady decrease to zero of the degree of approximation, $E_n(f) := \min_{P_n} \|f - P_n\|$:

$$E_{n+1}(f) \leq (1 - \varepsilon_n)E_n(f), \quad \text{with } \log(1/\varepsilon_n) = o(n). \tag{3.2}$$

Unfortunately, one does not know what structural properties of f ensure (3.2). It is true that the interesting paper of Fiedler and Jurkat [**3**] contains results of this type, but for the L_1—instead of the uniform—approximation.

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