DOUBLE COMMUTANTS OF OPERATORS
QUASI-SIMILAR TO NORMAL OPERATORS

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Abstract. It is shown that double commutants of operators quasi-similar to normal operators are reflexive.

An algebra $\mathcal{A}$ of bounded linear operators on a Hilbert space is said to be reflexive if $\mathcal{A} = \text{Alg Lat } \mathcal{A}$, where Lat $\mathcal{A}$ denotes the family of all subspaces invariant under all elements of $\mathcal{A}$ and Alg Lat $\mathcal{A}$ is the algebra of all operators $X$ for which $XM \subseteq M$ for every $M \in \text{Lat }\mathcal{A}$. If $\mathcal{A}$ is a von Neumann algebra, then $M \in \text{Lat }\mathcal{A}$ means that the projection to $M$ belongs to the commutant $\mathcal{A}'$, therefore Alg Lat $\mathcal{A}$ coincides with the double commutant $\mathcal{A}''$. Thus the von Neumann double commutant theorem can be rephrased as follows (cf. [4, Theorem 9.17]): every von Neumann algebra is reflexive.

Recall some definitions. For operators $T_1$ and $T_2$, let us write $T_1 \prec T_2$ (resp. $T_1 \prec T_2$) to denote that there exists a quasi-affinity, i.e. injection with dense range (resp. an injection) $X$ such that $XT_1 = T_2X$. $T_1$ and $T_2$ are quasi-similar by definition if $T_1 \prec T_2$ and $T_2 \prec T_1$.

If $N$ is a normal operator, the commutant $\{N\}'$ becomes a von Neumann algebra by the Fuglede theorem, so that both the commutant and the double commutant $\{N\}''$ are reflexive. If an operator $T$ is quasi-similar to a normal operator $N$, the commutant $\{T\}'$ is reflexive because the reflexivity of commutant is preserved under quasi-similarity [1] and $\{N\}'$ is reflexive. In this note we shall show that the double commutant $\{T\}''$ is reflexive too.

Theorem. If an operator $T$ is quasi-similar to a normal operator $N$, then the double commutant $\{T\}''$ is reflexive.

Before going into the proof, let us present an immediate consequence.

Corollary. If $T$ is a contraction of class $C_{11}$, that is, $\lim_{n \to \infty} \|T^n x\| \neq 0$ and $\lim_{n \to \infty} \|T^n x\| \neq 0$ for every nonzero $x$, then the double commutant $\{T\}''$ is reflexive.

In fact, it is well known that a contraction of class $C_{11}$ is quasi-similar to a unitary operator (see [5, Proposition II.3.5]).
This result for a $C_{11}$-contraction $T$ was proved in [3, Theorem 3] under the condition that every injection in $(T)'$ has dense range.

For the proof of the theorem we need a lemma, which is an analog of [3, Lemma 5] for $C_{11}$-contractions.

**LEMMA.** Let $T$ be an operator quasi-similar to a normal operator. Then, for $\mathcal{M} \in \text{Lat } T$ (i.e., the family of all $T$-invariant subspaces), the following conditions are mutually equivalent.

(i) $T|\mathcal{M}$ is quasi-similar to a normal operator.

(ii) $T|\mathcal{M} \succ N_1$ for some normal operator $N_1$.

(iii) $\mathcal{M} = (\text{ran } Z)'$ for some $Z \in \{T\}'$.

If one of these conditions holds, then $\mathcal{M} \in \text{Lat } \{T\}'$.

**Proof.** Let $X$ and $Y$ be quasi-affinities such that $XT = NX$ and $TY = YN$ for some normal operator $N$ on $\mathcal{H}$. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Since $N_1 < T|\mathcal{M} < N(X\mathcal{M})^{-}$, it follows from [2, Lemma 4.1] that $(X\mathcal{M})^{-}$ is $N$-reducing and $N(X\mathcal{M})^{-}$ is unitarily equivalent to $N_1$. Therefore we have $(T|\mathcal{M})^* < N_1^* < N^*(X\mathcal{M})^{-} < N^* < T^*$. Denote by $W$ the injection from $\mathcal{M}$ to $\mathcal{H}$ (the space on which $T$ acts) such that $W(T|\mathcal{M})^* = T^*W$, and by $J$ the inclusion mapping of $\mathcal{M}$ into $\mathcal{H}$. Then $Z = JW^*$ is an operator required in (iii).

(iii) $\Rightarrow$ (i): Define an operator $K$ from $\mathcal{H}$ to $\mathcal{M}$ by $Kx = ZYx$ for $x \in \mathcal{H}$. Then $K$ has dense range and $(T|\mathcal{M})K = KN$. Also $(X\mathcal{M})(T|\mathcal{M}) = N(X\mathcal{M})$ and $X\mathcal{M}$ is injective. Therefore it follows from [2, Lemma 4.1] that $N(\text{ker } K)^+$ and $N(X\mathcal{M})^{-}$ are unitarily equivalent normal operators, and so $T|\mathcal{M}$ is quasi-similar to a normal operator $N(X\mathcal{M})^{-}$. Finally it is clear that $\mathcal{M} \in \text{Lat } \{T\}'$ for any subspace $\mathcal{M}$ satisfying (iii).

**Proof of Theorem.** By assumption there are quasi-affinities $X$ and $Y$ satisfying $XT = NX$ and $TY = YN$. Without loss of generality, the product $XY$ can be assumed to be nonnegative. Indeed, since $XY \in \{N\}'$, $XY$ admits the polar factorization $XY = UP$ in the von Neumann algebra $\{N\}'$, where $U$ is unitary and $P$ is nonnegative. The quasi-affinities $U^*X$ and $Y$ satisfy the required condition.

The inclusion $\{T\}' \subseteq \text{Alg } \text{Lat } \{T\}'$ is obvious. To see the converse inclusion, take $A \in \text{Alg } \text{Lat } \{T\}'$. In order to show $A \in \{T\}'$, it suffices to prove that $A$ commutes with $YCX$ for all $C \in \{N\}'$. In fact, then $A$ commutes with $YX$, and also with $YXBYX$ for any $B \in \{T\}'$, because $XY$ belongs to $\{N\}'$. Therefore we have

$$YXBYX = YXBYXA = AYXBYX = YXABYX.$$  

Then the quasi-affinity properties of $X$ and $Y$ imply $BA = AB$, hence $A \in \{T\}'$.

Let us show that $A$ commutes with $YCX$ for all $C \in \{N\}'$. Since $\{N\}'$ is a von Neumann algebra, we have only to show that $A(YHX) = (YHX)A$ for any selfadjoint injection $H \in \{N\}'$. Take $\mathcal{M} \in \text{Lat } \{XYHX\}'$. Since $\{N\}' \subseteq \{XYHX\}'$, we have $\mathcal{M} \in \text{Lat } \{N\}'$, hence $\mathcal{M}$ is $N$-reducing and $N|\mathcal{M}$ is normal. Then, since $(Y\mathcal{M})^{-} \subseteq \text{Lat } T$ and $T(Y\mathcal{M})^{-} \succ N|\mathcal{M}$, it follows from the Lemma that $(Y\mathcal{M})^{-} \subseteq \text{Lat } \{T\}' \subseteq \text{Lat } A$, and so $XYHXAY \mathcal{M} \subseteq (XYHX\mathcal{M})^{-} \subseteq \mathcal{M}$. We can conclude that $XYHXAY \in \text{Alg } \text{Lat } \{XYHX\}'$. Then, since the commutant $\{XYHX\}'$ of the
selfadjoint operator $XYHXY$ is reflexive, $XYHXY \in \{ XYHXY \}'$. Taking $H = I$, we have $(XY)^2 XY A Y = XYA Y (XY)^2$ and $(XY)^2 X A Y = X A Y (XY)^2$ by the injectivity of $XY$. Since $XY$ is assumed to be nonnegative, taking the square root of $(XY)^2$, we obtain $XYA Y = XA Y XY$. Then for any selfadjoint injection $H$ we have

$$XYHXYXYHXY = XYHXYXYHXY = XYHXYXYHXY.$$  

Finally it follows from the injectivity of $XYHXY$ and the dense range property of $Y$ that $YHA Y = A Y H X$.

REFERENCES