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FINITELY ADDITIVE MEASURES ON N AND THE ADDITIVE PROPERTY

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ABSTRACT. Finitely additive measures on $\mathbf{N}$ satisfying an approximation of countable additivity, called (AP), are studied. These measures are generalizations of p-points. From a p-point a translation invariant measure with (AP) is constructed. It is consistent that no measure with (AP) exists.

0. Introduction. Finitely additive measures on ω $(= \mathbf{N} = \{0, 1, 2, \ldots\})$ which satisfy a weak form of countable additivity are studied. A *measure* will always mean a finitely additive $\mu\colon \mathcal{P}(\omega) \to [0,1]$ such that $\mu(\omega) = 1$ and $\mu(\{n\}) = 0$. (Nothing is gained by letting $\mu(\omega)$ take other values > 0.) Of course no measure is countably additive, but the following property is conceivable.

DEFINITION. A measure μ has the *additive property* (denoted (AP)) if for every disjoint collection $\{A_n\colon n < \omega\}$ of subsets of ω there is $A \subseteq \omega$ so that, for all n, $A_n \subseteq A$ (mod f.) and $\mu(A) = \sum \mu(A_n)$. ($A \subseteq B$ (mod f.) if $A \setminus B$ is finite. $A = B$ (mod f.) is defined similarly.)

Since $B \subseteq A$ (mod f.) implies $\mu(B) \leq \mu(A)$, (AP) is equivalent to: for every sequence $(A_n)_{n<\omega}$ of pairwise disjoint subsets of ω there is $(A_n^*)_{n<\omega}$ such that, for all n, $A_n = A_n^*$ (mod f.) and $\mu(\bigcup A_n^*) = \sum \mu(A_n^*)$. The additive property has been studied (for not necessarily total measures) in [**B** and **F**].

EXAMPLE. Any ultrafilter on ω can be identified with a measure; i.e. sets in the ultrafilter have measure 1; other sets have measure 0. An ultrafilter with (AP) is called a p-point. p-points were introduced and studied by Walter Rudin [**R**] in the context of $\beta\mathbf{N} - \mathbf{N}$. Since p-points have been extensively studied, there is a lot of information about some measures with (AP). Under the assumption of CH or, more generally, MA, a p-point exists (cf. [**J**, pp. 257–259]). However, Shelah [**W**, **S**] has shown it is consistent, assuming the consistency of ZFC, that no p-point exists.

From some points of view, ultrafilters are unsatisfactory measures. Following Maharam [**M**], call a measure a *density* if it extends natural asymptotic density; i.e. $\mu(A) = \lim_{n\to\infty} |A \cap n|/n$ whenever this limit exists. (Here $n = \{0, 1, \ldots, n-1\}$.) A measure is *translation invariant* if $\mu(A) = \mu(A + a)$ for all $A \in \mathcal{P}(\omega)$ and $a \in \mathbf{Z}$. (Here $n \in A + a$ iff $n - a \in A$.) The first section of the paper is devoted to constructing a translation invariant density from a p-point. The route taken is to construct a linear functional, show it has (AP), and then smooth it out.

The second section contains a proof that it is consistent that no measure has (AP). This proof is a generalization and simplification of Shelah's [**S**] for the nonex-

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istence of p-points. I do not know if the existence of a measure with (AP) implies there is a p-point.

I would like to thank A. R. Freedman and John Sember for asking me about measures with (AP) and answering my many naïve questions.

1. From p-points to densities. As well as measures with (AP), it is convenient to consider measures with (AP0), where (AP0) is defined just as (AP) was but the A_n's are required to have measure 0. Consider the Banach space $l^\infty = \{x\colon \omega \to \mathbf{R}\colon x \text{ is bounded in absolute value}\}$. Call a linear functional $F\colon l^\infty \to \mathbf{R}$ *positive* if (1) for all n, $x(n) \geq 0$ implies $F(x) \geq 0$; (2) $F(\chi_\omega) = 1$; and (3) $F(\chi_{\{n\}}) = 0$ for all n. As usual, χ_A is the characteristic function of A. Of course any positive linear functional F induces a measure μ_F by $\mu_F(A) = F(\chi_A)$. Also any measure μ induces a positive linear functional F_μ by $F_\mu(x) = \int x\mu$. (Here

$$\int x\mu = \lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \frac{k}{n}\mu\left(x^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)\right).)$$

It is standard and easy to see that integration theory works for l^∞ and measures (in the sense of this paper). The Riesz representation theorem explains why the positive linear functionals were singled out.

THEOREM. *A linear functional F (on l^∞) is positive iff there is a measure μ such that $F(x) = \int x\mu$ for all x.*

DEFINITION. Suppose F is a positive linear functional (on l^∞). Temporarily call a sequence $(x^n)_{n<\omega}$ of elements of l^∞ *positive convergent* if, for all m and n, $x^n(m) \geq 0$ and the pointwise sum $\sum x^n \in l^\infty$. Then F has (AP) if for each positive convergent sequence $(x^n)_{n<\omega}$ there exists $(y^n)_{n<\omega}$ so that: for all n, $y^n = x^n$ (mod f.); for all n and m, $y^n(m) \geq 0$; and $F(\sum y^n) = \sum F(y^n)$ $(= \sum F(x^n))$. One can similarly define (AP0) for positive linear functionals.

1.1. PROPOSITION. *If a measure μ has* (AP) ((AP0)), *then F_μ has* (AP) ((AP0)).

PROOF. Let F denote F_μ. Suppose $(x^n)_{n<\omega}$ is as in the definition of (AP) for functionals. Choose a natural number N which is an upper bound for $\sum x^n$. Fix $\varepsilon > 0$. Now a sequence $(y^n)_{n<\omega}$ is constructed so that: $y^n = x^n$ (mod f.); for all n and m, $y^n(m) \geq 0$; and $F(\sum y^n) < \sum F(y^n) + \varepsilon$. Let $z^n = \sum_{i\leq n} x^i$. For all real $r < N$, define A_r^n to be $\{m\colon z^n(m) > r\}$. Note: for all n, $A_r^n \subseteq A_r^{n+1}$. By the (AP) of μ we can find $A_r \supseteq A_r^n$ (mod f.) for all n and $\mu(A_r) = \lim \mu(A_r^n)$. Suppose $\varepsilon > 1/lN$. Let $R = \{i/lN\colon 0 \leq i < lN^2 - 1\}$. Let $y^n(m) = 0$ if for some $r \in R$, $m \in A_r^n - A_r$. Otherwise let $y^n(m) = x^n(m)$.

Claim. $(\sum y^n)(m) > r$ implies $m \in A_r$ (for $r \in R$).

PROOF (OF CLAIM). First note $(\sum y^n)(m) > r$ implies $(\sum x^n)(m) > r$. Suppose $m \notin A_r$. Let n_0 be the least n so that $(\sum_{i\leq n} x^i)(m) > r$. So for all $i \geq n_0$, $m \in A_r^i \setminus A_r$. Hence, for all $i \geq n_0$, $y^i(m) = 0$. Hence $(\sum y^n)(m) \leq r$.

It can be supposed that $A_0 = \omega$ and $A_r \supset A_{r'}$ for $r < r' \in R$. Let $B_r = A_r \setminus A_{r+1/lN}$ (where $A_N = \varnothing$). By the claim for $m \in B_r$, $(\sum y^n)(m) \leq r + \varepsilon$. Since $\omega = \bigcup_{r\in R} B_r$,

$$F\left(\sum y^n\right) \leq \sum_{r\in R} (r+\varepsilon)\mu(B_r).$$

Also, for all n,

$$\sum_{k=0}^{n} F(y^k) = F\left(\sum_{k=0}^{n} y^k\right) \geq \sum_{r\in R} r\mu\left[(A_r^n \setminus A_{r+1/lN}^n)\bigcap A_r\right],$$

whence $\sum F(y^n) \geq \sum_{r\in R} r\mu(B_r)$. So $F(\sum y^n) \leq \sum F(y^n) + \varepsilon$.

To finish the proof, choose sequences $(y_m^n)_{n<\omega}$ $(1 \leq m < \omega)$ so that: $y_m^n = x^n$ (mod f.) for all n, m; for all i, n and $m < m'$, $y_m^n(i) \geq y_{m'}^n(i) \geq 0$; and for all m, $F(\sum y_m^n) < \sum F(y_m^n) + 1/m$. So $(y_n^n)_{n<\omega}$ is the desired sequence. Exactly the same proof works for (AP0).

Although a positive linear functional with (AP) has been constructed from a measure with (AP), it still must be "smoothed out". Let

$$C = \begin{bmatrix} 1 & & 0 & \dots \\ 1/2 & 1/2 & 0 & \dots \\ \vdots & & & \\ 1/n & 1/n & 1/n & 0\dots \\ \vdots & \vdots & & \end{bmatrix}$$

be the Cesàro matrix. C is a transformation on l^∞; i.e.

$$C(x)(n) = \frac{1}{(n+1)} \sum_{i<n+1} x(i).$$

1.2. LEMMA. *Suppose F is a positive linear functional. Then $F \circ C$ is a translation invariant positive linear functional such that, for all $A \subseteq \omega$, if $\lim |A \cap n|/n = d$ then $F \circ C(\chi_A) = d$. Further, if F has* (AP) ((AP0)) *so does $F \circ C$.*

PROOF. The first two properties result from the fact that $F(z) = 0$ whenever z has limit 0. It remains to see why $F \circ C$ has (AP) if F does. Assume F has (AP). Note, given $(x^n)_{n<\omega}$ as in the definition of (AP) the required $(y^n)_{n<\omega}$ can always be chosen so that for all n there is N such that

$$y^n(m) = \begin{cases} 0 & \text{if } m < N, \\ x^n(m) & \text{otherwise.} \end{cases}$$

Temporarily call such a y^n, x^n above N. Note that for all $x \in l^\infty$ and $N < \omega$, if y is x above N then $F \circ C(x) = F \circ C(y)$. To see this let $z = C(x) - C(y)$. Since $\lim z(m) = 0$, $F(z) = 0$.

Suppose now $(x^n)_{n<\omega}$ are as in the definition of (AP). Let $z^n = C(x^n)$. Choose $(N_n)_{n<\omega}$ so that if w^n is z^n above N_n then $F(\sum w^n) = \sum F(w^n)$. Let y^n be x^n above N_n. Note that for all n and m, $C(y^n)(m) < w^n(m)$. So

$$F \circ C\left(\sum y^n\right) = F\left(\sum C(y^n)\right) \leq F\left(\sum w^n\right) = \sum F \circ C(y^n).$$

Since $\sum F \circ C(y^n)$ is always $\leq F \circ C(\sum y^n)$, $(y^n)_{n<\omega}$ is as required to show $F \circ C$ satisfies (AP).

1.3. THEOREM. *If there is a measure with* (AP) ((AP0)) *then there is a translation invariant density with* (AP) ((AP0)).

Although I cannot show that the existence of a measure with (AP) implies the existence of a p-point, if some ultrafilter induces a density with (AP0), then there is a p-point.

1.4. THEOREM. *Suppose μ is an ultrafilter and $F_\mu \circ C$ has* (AP0). *Then there is a p-point.*

PROOF. Choose $0 = \alpha_0 < \beta_0 < \alpha_1 < \cdots$ so that $\alpha_n > n\beta_{n-1}$ and $\beta_n > n\alpha_n$. Then $(i-\beta_{n-1})/i > 1-1/n$ ($(i-\alpha_n)/i > 1-1/n$) and $\beta_{n-1}/i < 1/n$ ($\alpha_n/i < 1/n$) for $\alpha_n \le i < \beta_n$ ($\beta_n \le i < \alpha_{n+1}$). Either $\bigcup_{n>0}[\alpha_n, \beta_n) \in \mu$ or $\bigcup_{n>0}[\beta_n, \alpha_{n+1}) \in \mu$. Assume $X = \bigcup_{n>0}[\alpha_n, \beta_n) \in \mu$. Define $f\colon \omega \to \omega$ so that

$$f(i) = \begin{cases} n & \text{if } \alpha_n \le i < \beta_n, \\ 0 & \text{if } i \notin X. \end{cases}$$

Define an ultrafilter V on ω by $A \in V$ iff $f^{-1}(A) \in \mu$.

Claim. V is a p-point.

PROOF (OF CLAIM). To obtain a contradiction, suppose $(A_n)_{n<\omega}$ is a collection of disjoint sets such that, for all n, $A_n \notin V$, but for any $A \in V$ there is n so that $A \cap A_n$ is infinite. Choose B_n so that for $i \in [\beta_{m-1}, \beta_m)$, $i \in B_n$ iff $m \in A_n$. So for $i \in [\alpha_k, \beta_k)$ and $k \notin A_n$, $C(\chi_{B_n})(i) = |B_n \cap i|/i < 1/k$. Since $\bigcup_{k \notin A_n}[\alpha_k, \beta_k) \in \mu$, $F_\mu \circ C(\chi_{B_n}) = 0$. Choose B so that: $F_\mu \circ C(\chi_B) = 0$; and for all n, $B \supseteq B_n$ (mod f.). (Such a B exists by (AP0) of $F_\mu \circ C$.) Choose $Y \in \mu$ so that $Y \subseteq X$ and for all $i \in Y$, $|B \cap i|/i < 1/2$. By the choice of $(A_n)_{n<\omega}$ there exists an n so that $Y \cap f^{-1}(A_n)$ is infinite. Choose m so that $B_n \subseteq B \cup [0, \beta_{m-1})$. Pick $k > 2$, m and i so that $i \in f^{-1}(A_n) \cap Y \cap [\alpha_k, \beta_k)$. Since $B \supseteq [\beta_{k-1}, \beta_k)$, $|B \cap i|/i \ge (i - \beta_{k-1})/i > 1/2$. This contradicts the choice of Y.

1.5. EXAMPLE. Suppose μ is a p-point. Choose an increasing sequence $k(n)$ so that $n^2/k(n) < 1/n$. Define ν by $A \in \nu$ iff $\{i\colon \{n\colon k(n) + i \in A\} \in \mu\} \in \mu$. It is not hard to show $F_\nu \circ C$ has (AP).

Let μ be any nonprincipal ultrafilter and let $(\beta_n)_{n<\omega}$ be as in Theorem 1.4. Construe $\mu \times \mu$ (defined by $A \in \mu \times \mu$ iff $\{n\colon \{m\colon (n,m) \in A\} \in A\} \in \mu$) as an ultrafilter μ' on ω. Let ν be the ultrafilter containing all sets of the form $\{\beta_n\colon n \in A\}$ for some $A \in \mu'$. Then $F_\nu \circ C$ does not have (AP0).

2. There may be no measure with (AP). This section depends on some of the elements of Shelah's proof [**S**] that there may be no p-points. A poset $\mathbf{P}$ has the *ω^ω-bounding* property if, whenever G is $\mathbf{P}$-generic and $h \in V[G]$ is a function from ω to ω, there is $g\colon \omega \to \omega$ so that $g \in V$ and $h < g$ (i.e. for all n, $h(n) < g(n)$). It is shown [**S**, V.4] that an iteration with countable support of ω-proper posets with the ω^ω-bounding property is itself ω-proper and has the ω^ω-bounding property. (It is not necessary to know the definition of ω-proper to understand this paper.)

A filter F is called a *P-filter* iff it is nonprincipal and for any $\{A_n\colon n < \omega\}$ of elements of I, the dual ideal, there is $A \in I$ so that for all n, $A_n \subseteq A$ (mod f.). Further, F is *fat* if, given $\{w_n\colon n < \omega\}$, a set of disjoint finite subsets of ω, there is an infinite $S \subseteq \omega$ so that $\bigcup_{n \in S} w_n \in I$. Shelah shows if F is a fat P-filter then $\mathbf{P}(F)^\omega$ is an ω-proper poset with the ω^ω-bounding property. Here

$\mathbf{P}(F) = \{f\colon f\colon A \to 2$ for some $A \in I\}$ ordered by containment. So forcing with $\mathbf{P}(F)$ introduces a subset of ω.

The following lemma is the analogoue of VI.4.7 in [**S**].

2.1. LEMMA. *Suppose F is a fat P-filter and $\mathbf{P} = \mathbf{P}(F)^\omega * \tilde{Q}$ has the ω^ω-bounding property. Then $\mathbf{P} \Vdash$ "F cannot be extended to a measure with* (AP)" (*i.e. there is no measure μ with* (AP) *such that $\mu(A) = 1$ for all $A \in F$*).

PROOF. Assume G is $\mathbf{P}$-generic and μ is a measure with (AP) extending F (in $V[G]$). Forcing with $P(F)^\omega$ introduces a sequence $\langle A_0, A_1, \ldots\rangle$ of subsets of ω defined by $n \in A_i$ iff there is $\langle f_0, f_1, \ldots\rangle \in G$ so that $f_i(n) = 1$. There are two cases. Either for all n and $\varepsilon > 0$ there is $k > n$ so that $\mu(\bigcup_{n\le i<k} -A_i) > 1 - \varepsilon$ ($-A_i$ denotes $\omega\backslash A_i$), or there is n and $\delta > 0$ so that, for all $k > n$, $\mu(\bigcap_{n\le i<k} A_i) > \delta$. The two cases are similar, so I will only do the first one, which is the more difficult.

Assume the first case holds. Choose $g \in V$ so that for all n, $\mu(\bigcup_{n\le i<g(n)} -A_i) > 1 - 1/2^{n+2}$. (That such a $g \in V[G]$ exists is implied since the first case has been assumed. The ω^ω-bounding property guarantees g can be chosen in V.) Let $B_n = \bigcup -A_i(g^{(n)}(0) \le i < g^{(n+1)}(0))$. Here $g^{(n)}$ denotes the nth iterate of g. The B_n's have been chosen so that $\mu(B_n) > 1 - 1/2^{n+2}$. If μ were countably additive, then $\mu(\bigcap B_n)$ would be $> 1/2$. But (AP) implies there is h (which can be taken in V) so that $\mu(\bigcap(B_n \cup [0, h(n))))) > 1/2$.

Let $p = (\langle f_0, f_1, \ldots\rangle, \tilde{q})$ be a condition which forces the above. Since F is a P-filter and $\operatorname{dom} f_i \in I$, the dual ideal, there is $D \in I$ so that for all i, $\operatorname{dom} f_i \subseteq D$ (mod f.). Choose a strictly increasing sequence α_n ($n \in \omega$) so that: for all n, $\alpha_n > h(n)$; and for all $i < g^{(n+1)}(0)$, $\operatorname{dom} f_i \cap [\alpha_n, \infty] \subseteq D$. If $g^{(n)}(0) \le i < g^{(n+1)}(0)$, define $f_i' = f_i \cup 1_{[\alpha_n, \alpha_{n+1}]\backslash D}$. Here 1_X denotes the function with domain X which is constantly 1. The choice of $(\alpha_n)_{n<\omega}$ guarantees f_i' is a function. So $r = \langle f_0', f_1', \ldots\rangle \in \mathbf{P}(F)^\omega$. Note: if $g^{(n)}(0) \le i < g^{(n+1)}(0)$, then

$$r \Vdash \tilde{A}_i \supseteq [\alpha_n, \alpha_{n+1}) \setminus D.$$

($\tilde{A}_i$ is a name for A_i, etc.) So

$$r \Vdash (\tilde{B}_n \cup [0, h(n))) \cap [\alpha_n, \alpha_{n+1}) = \tilde{B}_n \cap [\alpha_n, \alpha_{n+1}) \subseteq D.$$

Hence,

$$r \Vdash \bigcap(\tilde{B}_n \cup [0, h(n))) \subseteq [0, \alpha_0) \cup^{\cdot} D \in I.$$

So

$$r \Vdash \mu\left(\bigcap(\tilde{B}_n \cup [0, h(n)))\right) = 0.$$

But r and p are consistent, a contradiction.

2.2. THEOREM. *Assume* ZF *is consistent. It is consistent with* ZFC *that no measure on ω has* (AP).

PROOF. Beginning with $V \models$ GCH, using Lemma 2.1 and a countable support iteration of ω-proper posets with the ω^ω-bounding property, one can show the following statement is consistent with ZFC:

> if $F \subset \mathcal{P}(\omega)$ is a filter base of cardinality ω_1 for a fat P-filter, then F cannot be extended to a measure with (AP).

So the theorem reduces to verifying the following claim.

Claim. Suppose $V \models \mathrm{CH}$, $\mathbf{P}$ has the ω^ω-bounding property and G is $\mathbf{P}$ generic. If $\mu \in V[G]$ is a measure with (AP) then there is $F \subseteq \{A\colon \mu(A) = 1\}$, a filter base of cardinality ω_1 for a fat P-filter.

PROOF (OF CLAIM). First note: $\{A\colon \mu(A) = 1\}$ is a fat filter. If $\{B_i\colon i < n+1\}$ are disjoint subsets of ω then there is some i so that $\mu(B_i) < 1/n$. Suppose $\{w_n\colon n < \omega\}$ is a collection of disjoint finite subsets of ω. Choose a sequence $S_1 \supseteq S_1 \supseteq \cdots$ of infinite sets so that $\mu(\bigcup_{i\in S_n} w_i) < 1/n$. Choose an infinite S so that for all n, $S \subseteq S_n$ (mod f.). So for all n,

$$\mu\left(\bigcup_{i\in S} w_i\right) \le \mu\left(\bigcup_{i\in S\setminus S_n} w_i\right) + \mu\left(\bigcup_{i\in S_n} w_i\right) \le 0 + \frac{1}{n}.$$

Hence $\mu(\bigcup_{i\in S} w_i) = 0$.

Now F, or more exactly a base for the dual ideal, will be defined inductively. Choose $\{A_\alpha\colon \alpha < \omega_1\}$ so that: $\mu(A_\alpha) = 0$; and if $\{w_n\colon n \in \omega\} \in V$ is a collection of disjoint finite subsets of ω then there is α and an infinite $S \subseteq \omega$ so that $\bigcup_{n\in S} w_n \subseteq A_\alpha$. Choose countable I_α $(\alpha < \omega_1) \subseteq \{A\colon \mu(A) = 0\}$ inductively so that for all $\alpha, \beta < \omega_1$, $\alpha < \beta$ implies $I_\alpha \subset I_\beta$; I_α is closed under finite unions; $A_\alpha \in I_\alpha$; and there is $A \in I_{\alpha+1}$ so that for all $B \in I_\alpha$, $A \supset B$ (mod f.). Let $I = \bigcup_{\alpha<\omega_1} I_\alpha$ and F be the dual filter base of I.

It is easy to see F generates a P-filter. Suppose $\{w_n\colon n \in \omega\}$ is a collection of disjoint finite subsets of ω. By the ω^ω-bounding property of $\mathbf{P}$ there is $g \in V$ so that for all i there is n such that $[i, g(i)) \supseteq w_n$. Let $u_n = [g^{(n)}(0),\ g^{(n+1)}(0))$. By the choice of I there is an infinite S and $\alpha < \omega_1$ so that $\bigcup_{n\in S} u_n \subseteq A_\alpha$. So $S' = \{n\colon w_n \subseteq A_n\}$ shows F generates a fat filter.

2.3. REMARK. The above proof shows that if μ is a measure, then $I = \{A\colon \mu(A) = 0\}$ is a quasi-σ-ring. That is, if $\{A_n\colon n < \omega\}$ is a collection of disjoint elements of I, then there is an infinite S so that $\bigcup_{n\in S} A_n \in I$.

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