FINITELY ADDITIVE MEASURES ON $\mathbb{N}$
AND THE ADDITIVE PROPERTY

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ABSTRACT. Finitely additive measures on $\mathbb{N}$ satisfying an approximation of countable additivity, called (AP), are studied. These measures are generalizations of $p$-points. From a $p$-point a translation invariant measure with (AP) is constructed. It is consistent that no measure with (AP) exists.

0. Introduction. Finitely additive measures on $\omega (= \mathbb{N} = \{0, 1, 2, \ldots \})$ which satisfy a weak form of countable additivity are studied. A measure will always mean a finitely additive $\mu: \mathcal{P}(\omega) \to [0, 1]$ such that $\mu(\omega) = 1$ and $\mu(\{n\}) = 0$. (Nothing is gained by letting $\mu(\omega)$ take other values $> 0$.) Of course no measure is countably additive, but the following property is conceivable.

DEFINITION. A measure $\mu$ has the additive property (denoted (AP)) if for every disjoint collection $\{A_n: n < \omega\}$ of subsets of $\omega$ there is $A \subseteq \omega$ so that, for all $n$, $A_n \subseteq A \pmod{f.}$ and $\mu(A) = \sum \mu(A_n)$. ($A \subseteq B \pmod{f.}$ if $A \setminus B$ is finite. $A = B \pmod{f.}$ is defined similarly.)

Since $B \subseteq A \pmod{f.}$ implies $\mu(B) \leq \mu(A)$, (AP) is equivalent to: for every sequence $(A_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ there is $(A^*_n)_{n<\omega}$ such that, for all $n$, $A_n = A^*_n \pmod{f.}$ and $\mu(\bigcup A^*_n) = \sum \mu(A^*_n)$. The additive property has been studied (for not necessarily total measures) in \cite{B}.

EXAMPLE. Any ultrafilter on $\omega$ can be identified with a measure; i.e. sets in the ultrafilter have measure 1; other sets have measure 0. An ultrafilter with (AP) is called a $p$-point. $p$-points were introduced and studied by Walter Rudin \cite{R} in the context of $\beta\mathbb{N} - \mathbb{N}$. Since $p$-points have been extensively studied, there is a lot of information about some measures with (AP). Under the assumption of CH or, more generally, MA, a $p$-point exists (cf. \cite{J}, pp. 257–259). However, Shelah \cite{W, S} has shown it is consistent, assuming the consistency of ZFC, that no $p$-point exists.

From some points of view, ultrafilters are unsatisfactory measures. Following Maharam \cite{M}, call a measure a density if it extends natural asymptotic density; i.e. $\mu(A) = \lim_{n \to \infty} |A \cap n| / n$ whenever this limit exists. (Here $n = \{0, 1, \ldots, n-1\}$.) A measure is translation invariant if $\mu(A) = \mu(A + a)$ for all $A \in \mathcal{P}(\omega)$ and $a \in \mathbb{Z}$. (Here $n \in A + a$ iff $n - a \in A$.) The first section of the paper is devoted to constructing a translation invariant density from a $p$-point. The route taken is to construct a linear functional, show it has (AP), and then smooth it out.

The second section contains a proof that it is consistent that no measure has (AP). This proof is a generalization and simplification of Shelah’s \cite{S} for the nonex-
istence of p-points. I do not know if the existence of a measure with (AP) implies there is a p-point.

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1. From p-points to densities. As well as measures with (AP), it is convenient to consider measures with (AP0), where (AP0) is defined just as (AP) was but the \( A_n \)'s are required to have measure 0. Consider the Banach space \( l^\infty = \{ x : \omega \to \mathbb{R} : x \text{ is bounded in absolute value} \} \). Call a linear functional \( F : l^\infty \to \mathbb{R} \) positive if (1) for all \( n, x(n) \geq 0 \) implies \( F(x) \geq 0 \); (2) \( F(\chi_A) = 1 \); and (3) \( F(\chi_{\{n\}}) = 0 \) for all \( n \). As usual, \( \chi_A \) is the characteristic function of \( A \). Of course any positive linear functional \( F \) induces a measure \( \mu_F \) by \( \mu_F(A) = F(\chi_A) \). Also any measure \( \mu \) induces a positive linear functional \( F_\mu \) by \( F_\mu(x) = \int x \mu \). (Here

\[
\int x \mu = \lim_{n \to \infty} \sum_{k=-n^2}^{n^2} \frac{k}{n} \mu \left( x^{-1}\left( \left[ \frac{k}{n}, \frac{k+1}{n} \right) \right) \right).
\]

It is standard and easy to see that integration theory works for \( l^\infty \) and measures (in the sense of this paper). The Riesz representation theorem explains why the positive linear functionals were singled out.

**Theorem.** A linear functional \( F \) (on \( l^\infty \)) is positive iff there is a measure \( \mu \) such that \( F(x) = \int x \mu \) for all \( x \).

**Definition.** Suppose \( F \) is a positive linear functional (on \( l^\infty \)). Temporarily call a sequence \( (x^n)_{n<\omega} \) of elements of \( l^\infty \) positive convergent if, for all \( m \) and \( n \), \( x^n(m) \geq 0 \) and the pointwise sum \( \sum x^n \in l^\infty \). Then \( F \) has (AP) if for each positive convergent sequence \( (x^n)_{n<\omega} \) there exists \( (y^n)_{n<\omega} \) so that: for all \( n \), \( y^n = x^n \) (mod f.); for all \( n \) and \( m \), \( y^n(m) \geq 0 \); and \( F(\sum y^n) = \sum F(y^n) \) (\( = \sum F(x^n) \)). One can similarly define (AP0) for positive linear functionals.

1.1. **Proposition.** If a measure \( \mu \) has (AP) ((AP0)), then \( F_\mu \) has (AP) ((AP0)).

**Proof.** Let \( F \) denote \( F_\mu \). Suppose \( (x^n)_{n<\omega} \) is as in the definition of (AP) for functionals. Choose a natural number \( N \) which is an upper bound for \( \sum x^n \). Fix \( \epsilon > 0 \). Now a sequence \( (y^n)_{n<\omega} \) is constructed so that: \( y^n = x^n \) (mod f.); for all \( n \) and \( m \), \( y^n(m) \geq 0 \); and \( F(\sum y^n) < \sum F(y^n) + \epsilon \). Let \( z^n = \sum_{i \leq n} x^i \). For all real \( r < N \), define \( A_r^\infty \) to be \( \{ m : z^n(m) > r \} \). Note: for all \( n \), \( A_r^n \subseteq A_r^{n+1} \). By the (AP) of \( \mu \) we can find \( A_r \supseteq A_r^n \) (mod f.) for all \( n \) and \( \mu(A_r) = \lim \mu(A_r^n) \). Suppose \( \epsilon > 1/\lfloor N \rfloor \). Let \( R = \{ i/\lfloor N \rfloor : 0 \leq i < \lfloor N \rfloor^2 - 1 \} \). Let \( y^n(m) = 0 \) if for some \( r \in R \), \( m \in A_r^n \setminus A_r^\infty \). Otherwise let \( y^n(m) = x^n(m) \).

Claim. \( (\sum y^n)(m) > r \) implies \( m \in A_r \) (for \( r \in R \)).

**Proof (of Claim).** First note \( (\sum y^n)(m) > r \) implies \( (\sum x^n)(m) > r \). Suppose \( m \notin A_r \). Let \( n_0 \) be the least \( n \) so that \( (\sum x^i)(m) > r \). So for all \( i \geq n_0 \), \( m \in A_r^i \setminus A_r \). Hence, for all \( i \geq n_0 \), \( y^i(m) = 0 \). Hence \( (\sum y^n)(m) \leq r \).

It can be supposed that \( A_0 = \omega \) and \( A_r \supset A_{r'} \) for \( r < r' \in R \). Let \( B_r = A_r \setminus A_{r+1/\lfloor N \rfloor} \) (where \( A_N = \emptyset \)). By the claim for \( m \in B_r \), \( (\sum y^n)(m) \leq r + \epsilon \). Since \( \omega = \bigcup_{r \in R} B_r \),

\[
F \left( \sum y^n \right) \leq \sum_{r \in R} (r + \epsilon) \mu(B_r).
\]
Also, for all \( n \),
\[
\sum_{k=0}^{n} F(y^k) = F \left( \sum_{k=0}^{n} y^k \right) \geq \sum_{r \in R} r \mu \left( (A^r_n \setminus A^r_{n+1/\mu}) \cap A_r \right),
\]
whence \( \sum_{r \in R} r \mu (B_r) \). So \( F(\sum y^n) \leq \sum F(y^n) + \epsilon. \)

To finish the proof, choose sequences \( (y^m_m)_{m<\omega} \) for all \( n \), \( m \), \( n \), and \( m \), \( n \), and \( m \), \( m \), \( n \), \( m \), \( n \) \( (i) \geq 0 \); and for all \( m \), \( m \), \( (i) \). So \( (y^n_n)_{n<\omega} \) is the desired sequence. Exactly the same proof works for \( (\text{AP}) \).

Although a positive linear functional with \( (\text{AP}) \) has been constructed from a measure with \( (\text{AP}) \), it still must be "smoothed out". Let
\[
C = \begin{bmatrix}
1 & 0 & \cdots \\
1/2 & 1/2 & 0 & \cdots \\
\vdots \\
1/n & 1/n & 1/n & 0 & \cdots \\
\vdots & \vdots 
\end{bmatrix}
\]
be the Cesàro matrix. \( C \) is a transformation on \( l^\infty \); i.e.
\[
C(x)(n) = \frac{1}{n+1} \sum_{i<n+1} x(i).
\]

1.2. LEMMA. Suppose \( F \) is a positive linear functional. Then \( F \circ C \) is a translation invariant positive linear functional such that, for all \( A \subseteq \omega \), if \( \lim |A \cap n/n = d \) then \( F \circ C(x_A) = d \). Further, if \( F \) has \( (\text{AP}) \) \( (\text{AP}0) \) so does \( F \circ C \).

PROOF. The first two properties result from the fact that \( F(z) = 0 \) whenever \( z \) has limit 0. It remains to see why \( F \circ C \) has \( (\text{AP}) \) if \( F \) does. Assume \( F \) has \( (\text{AP}) \). Note, given \( (x^n_n)_{n<\omega} \) as in the definition of \( (\text{AP}) \) the required \( (y^n_n)_{n<\omega} \) can always be chosen so that for all \( n \) there is \( N \) such that
\[
y^n(m) = \begin{cases}
0 & \text{if } m < N, \\
x^n(m) & \text{otherwise}.
\end{cases}
\]
Temporarily call such a \( y^n \), \( x^n \) above \( N \). Note that for all \( x \in l^\infty \) and \( N < \omega \), if \( y \) is \( x \) above \( N \) then \( F \circ C(x) = F \circ C(y) \). To see this let \( z = C(x) - C(y) \). Since \( \lim z(m) = 0, F(z) = 0 \).

Suppose now \( (x^n_n)_{n<\omega} \) are as in the definition of \( (\text{AP}) \). Let \( z^n = C(x^n) \). Choose \( (N_n)_{n<\omega} \) so that if \( w^n \) is \( z^n \) above \( N_n \) then \( F(\sum w^n) = \sum F(w^n) \). Let \( y^n \) be \( x^n \) above \( N_n \). Note that for all \( n \) and \( m \), \( C(y^n)(m) < w^n(m) \). So
\[
F \circ C \left( \sum y^n \right) = F \left( \sum C(y^n) \right) \leq F \left( \sum w^n \right) = \sum F \circ C(y^n).
\]
Since \( \sum F \circ C(y^n) \) is always \( \leq F \circ C(\sum y^n) \), \( (y^n_n)_{n<\omega} \) is as required to show \( F \circ C \) satisfies \( (\text{AP}) \).
1.3. **Theorem.** If there is a measure with (AP) ((AP0)) then there is a translation invariant density with (AP) ((AP0)).

Although I cannot show that the existence of a measure with (AP) implies the existence of a p-point, if some ultrafilter induces a density with (AP0), then there is a p-point.

1.4. **Theorem.** Suppose \( \mu \) is an ultrafilter and \( F_\mu \circ C \) has (AP0). Then there is a p-point.

**Proof.** Choose \( 0 = \alpha_0 < \beta_0 < \alpha_1 < \cdots \) so that \( \alpha_n > n\beta_{n-1} \) and \( \beta_n > n\alpha_n \). Then \( (i - \beta_{n-1})/i > 1 - 1/n \) \((i - \alpha_n)/i > 1 - 1/n\) and \( \beta_{n-1}/i < 1/n \) \((\alpha_n/i < 1/n)\) for \( \alpha_n \leq i < \beta_n \). Either \( \bigcup_{n>0}[\alpha_n, \beta_n) \in \mu \) or \( \bigcup_{n>0}[\beta_n, \alpha_{n+1}) \in \mu \).

Assume \( X = \bigcup_{n>0}[\alpha_n, \beta_n) \in \mu \). Define \( f : \omega \to \omega \) so that

\[
 f(i) = \begin{cases} 
 n & \text{if } \alpha_n \leq i < \beta_n, \\
 0 & \text{if } i \notin X.
\end{cases}
\]

Define an ultrafilter \( V \) on \( \omega \) by \( A \in V \) iff \( f^{-1}(A) \in \mu \).

**Claim.** \( V \) is a p-point.

**Proof (of Claim).** To obtain a contradiction, suppose \( (A_n)_{n<\omega} \) is a collection of disjoint sets such that, for all \( n \), \( A_n \notin V \), but for any \( A \in V \) there is \( n \) so that \( A \cap A_n \) is infinite. Choose \( B_n \) so that for \( i \in [\beta_{m-1}, \beta_m) \), \( i \in B_n \) iff \( m \in A_n \). So for \( i \in [\alpha_k, \beta_k) \) and \( k \notin A_n \), \( C(xB_n)(i) = |B_n \cap i|/i < 1/k \). Since \( \bigcup_{k \notin A_n}[\alpha_k, \beta_k) \in \mu \), \( F_\mu \circ C(xB_n) = 0 \). Choose \( B \) so that \( F_\mu \circ C(xB) = 0 \); and for all \( n \), \( B \supseteq B_n \) (mod \( f \)). (Such a \( B \) exists by (AP0) of \( F_\mu \circ C \).) Choose \( Y \in \mu \) so that \( Y \subseteq X \) and for all \( i \in Y \), \( |B \cap i|/i < 1/2 \). By the choice of \( (A_n)_{n<\omega} \) there exists an \( n \) so that \( Y \cap f^{-1}(A_n) \) is infinite. Choose \( m \) so that \( B_n \subseteq B \cup [0, \beta_{m-1}) \). Pick \( k > 2 \), \( m \) and \( i \) so that \( i \in f^{-1}(A_n) \cap Y \cap [\alpha_k, \beta_k) \). Since \( B \supseteq [\beta_{k-1}, \beta_k) \), \( |B \cap i|/i \geq (i - \beta_{k-1})/i > 1/2 \). This contradicts the choice of \( Y \).

1.5. **Example.** Suppose \( \mu \) is a p-point. Choose an increasing sequence \( k(n) \) so that \( n^2/k(n) < 1/n \). Define \( \nu \) by \( A \in \nu \) iff \( \{i : \{n : k(n) + i \in A \} \in \mu \} \in \mu \). It is not hard to show \( F_\nu \circ \nu \) has (AP).

Let \( \mu \) be any nonprincipal ultrafilter and let \( (\beta_n)_{n<\omega} \) be as in Theorem 1.4. Construe \( \mu \times \mu \) (defined by \( A \in \mu \times \mu \) iff \( \{n : (m, n, m) \in A \} \in A \) \( \in \mu \)) as an ultrafilter \( \mu' \) on \( \omega \). Let \( \nu \) be the ultrafilter containing all sets of the form \( \{\beta_n : n \in A \} \) for some \( A \in \mu' \). Then \( F_\nu \circ \nu \) does not have (AP0).

2. There may be no measure with (AP). This section depends on some of the elements of Shelah’s proof [S] that there may be no p-points. A poset \( P \) has the \( \omega^\alpha \)-bounding property if, whenever \( G \) is \( P \)-generic and \( h \in V[G] \) is a function from \( \omega \) to \( \omega \), there is \( g : \omega \to \omega \) so that \( g \in V \) and \( h < g \) (i.e. for all \( n \), \( h(n) < g(n) \)). It is shown [S, V.4] that an iteration with countable support of \( \omega \)-proper posets with the \( \omega^\alpha \)-bounding property is itself \( \omega \)-proper and has the \( \omega^\alpha \)-bounding property. (It is not necessary to know the definition of \( \omega \)-proper to understand this paper.)

A filter \( F \) is called a \( P \)-filter if it is nonprincipal and for any \( \{A_n : n < \omega \} \) of elements of \( I \), the dual ideal, there is \( A \in I \) so that for all \( n \), \( A_n \subseteq A \) (mod \( f \)). Further, \( F \) is fat if, given \( \{w_n : n < \omega \} \), a set of disjoint finite subsets of \( \omega \), there is an infinite \( S \subseteq \omega \) so that \( \bigcup_{n \in S} w_n \in I \). Shelah shows if \( F \) is a fat \( P \)-filter then \( P(F)^\omega \) is an \( \omega \)-proper poset with the \( \omega^\alpha \)-bounding property. Here
\(P(F) = \{f : f : A \rightarrow 2 \text{ for some } A \in I\}\) ordered by containment. So forcing with \(P(F)\) introduces a subset of \(\omega\).

The following lemma is the analogue of VI.4.7 in [S].

**2.1. Lemma.** Suppose \(F\) is a fat \(P\)-filter and \(P = P(F)\times\tilde{Q}\) has the \(\omega^n\)-bounding property. Then \(P \Vdash \text{""} F\text{ cannot be extended to a measure with (AP)""} \) (i.e. there is no measure \(\mu\) with (AP) such that \(\mu(A) = 1\) for all \(A \in F\)).

**Proof.** Assume \(G\) is \(P\)-generic and \(\mu\) is a measure with (AP) extending \(F\) (in \(V[G]\)). Forcing with \(P(F)\times\tilde{Q}\) introduces a sequence \((A_0, A_1, \ldots)\) of subsets of \(\omega\) defined by \(n \in A_i\) iff there is \((f_0, f_1, \ldots) \in G\) so that \(f_i(n) = 1\). There are two cases. Either for all \(n\) and \(\varepsilon > 0\) there is \(k > n\) so that \(\mu(\bigcup_{n \leq i < k} -A_i) > 1 - \varepsilon\) (\(-A_i\) denotes \(\omega\setminus A_i\)), or there is \(n\) and \(\delta > 0\) so that, for all \(k > n\), \(\mu(\bigcap_{n \leq i < k} A_i) > \delta\). The two cases are similar, so I will only do the first one, which is the more difficult.

Assume the first case holds. Choose \(g \in V\) so that for all \(n\), \(\mu(\bigcup_{n \leq i < g(n)} -A_i) > 1 - 1/2^{n+2}\). (That such a \(g \in V[G]\) exists is implied since the first case has been assumed. The \(\omega^n\)-bounding property guarantees \(g\) can be chosen in \(V\).) Let \(B_n = \bigcup -A_i(g(n)(0) \leq i < g(n+1)(0))\). Here \(g(n)\) denotes the \(n\)th iterate of \(g\). The \(B_n\)'s have been chosen so that \(\mu(B_n) > 1 - 1/2^{n+2}\). If \(\mu\) were countably additive, then \(\mu(\bigcap B_n)\) would be \(> 1/2\). But (AP) implies there is \(h\) (which can be taken in \(V\)) so that \(\mu(\bigcap(B_n \cup [0, h(n)])) > 1/2\).

Let \(p = (f_0, f_1, \ldots, \tilde{q})\) be a condition which forces the above. Since \(F\) is a \(P\)-filter and \(\text{dom } f_i \subseteq D\) (mod \(\bar{f}\)). Choose a strictly increasing sequence \(\alpha_n (n \in \omega)\) so that: for all \(n\), \(\alpha_n > h(n)\); and for all \(i < g(n+1)(0)\), \(\text{dom } f_i \cap [\alpha_n, \infty] \subseteq D\). If \(g(n)(0) \leq i < g(n+1)(0)\), define \(f'_i = f_i \cup 1_{[\alpha_n, \alpha_{n+1}] \setminus D}\). Here \(1_X\) denotes the function with domain \(X\) which is constantly \(1\). The choice of \((\alpha_n)_{n<\omega}\) guarantees \(f'_i\) is a function. So \(r = (f'_0, f'_1, \ldots) \in P(F)\times\tilde{Q}\). Note: if \(g(n)(0) \leq i < g(n+1)(0)\), then

\[
\text{r} \Vdash A_i \supseteq [\alpha_n, \alpha_{n+1}] \setminus D.
\]

\((A_i\) is a name for \(A_i\), etc.) So

\[
r \Vdash (\exists n \in \omega : (B_n \cup [0, h(n)])) \cap [\alpha_n, \alpha_{n+1}] = B_n \cap [\alpha_n, \alpha_{n+1}] \subseteq D.
\]

Hence,

\[
r \Vdash \bigcap (\exists n \in \omega : (B_n \cup [0, h(n)])) \subseteq [0, \alpha_0) \cup D \in I.
\]

So

\[
r \Vdash \mu(\bigcap (\exists n \in \omega : (B_n \cup [0, h(n)]))) = 0.
\]

But \(r\) and \(p\) are consistent, a contradiction.

**2.2. Theorem.** Assume ZF is consistent. It is consistent with ZFC that no measure on \(\omega\) has (AP).

**Proof.** Beginning with \(V \models \text{GCH}\), using Lemma 2.1 and a countable support iteration of \(\omega\)-proper posets with the \(\omega^n\)-bounding property, one can show the following statement is consistent with ZFC:

if \(F \subseteq P(\omega)\) is a filter base of cardinality \(\omega_1\) for a fat \(P\)-filter, then \(F\) cannot be extended to a measure with (AP).
So the theorem reduces to verifying the following claim.

Claim. Suppose $V \models \text{CH}$, $\mathbf{P}$ has the $\omega^2$-bounding property and $G$ is a $\mathbf{P}$ generic. If $\mu \in V[G]$ is a measure with $(\text{AP})$ then there is $F \subseteq \{ A : \mu(A) = 1 \}$, a filter base of cardinality $\omega_1$ for a fat $\mathbf{P}$-filter.

Proof (of Claim). First note: $\{ A : \mu(A) = 1 \}$ is a fat filter. If $\{ B_i : i < n+1 \}$ are disjoint subsets of $\omega$ then there is some $i$ so that $\mu(B_i) < 1/n$. Suppose $\{ w_n : n < \omega \}$ is a collection of disjoint finite subsets of $\omega$. Choose a sequence $S_1 \supseteq S_2 \supseteq \cdots$ of infinite sets so that $\mu(\bigcup_{i \in S_n} w_i) < 1/n$. Choose an infinite $S$ so that for all $n$, $S \subseteq S_n (\text{mod } f)$. So for all $n$,

$$
\mu \left( \bigcup_{i \in S} w_i \right) \leq \mu \left( \bigcup_{i \in S \setminus S_n} w_i \right) + \mu \left( \bigcup_{i \in S_n} w_i \right) \leq 0 + \frac{1}{n}.
$$

Hence $\mu(\bigcup_{i \in S} w_i) = 0$.

Now $F$, or more exactly a base for the dual ideal, will be defined inductively. Choose $\{ A_\alpha : \alpha < \omega_1 \}$ so that: $\mu(A_\alpha) = 0$; and if $\{ w_n : n \in \omega \} \in V$ is a collection of disjoint finite subsets of $\omega$ then there is $\alpha$ and an infinite $S \subseteq \omega$ so that $\bigcup_{n \in S} w_n \subseteq A_\alpha$. Choose countable $I_\alpha (\alpha < \omega_1) \subseteq \{A : \mu(A) = 0\}$ inductively so that for all $\alpha, \beta < \omega_1$, $\alpha < \beta$ implies $I_\alpha \subset I_\beta$; $I_\alpha$ is closed under finite unions; $A_\alpha \in I_\alpha$; and there is $A \in I_{\alpha+1}$ so that for all $B \in I_\alpha$, $A \supseteq B (\text{mod } f)$. Let $I = \bigcup_{\alpha < \omega_1} I_\alpha$ and $F$ be the dual filter base of $I$.

It is easy to see $F$ generates a $\mathbf{P}$-filter. Suppose $\{ w_n : n \in \omega \}$ is a collection of disjoint finite subsets of $\omega$. By the $\omega^2$-bounding property of $\mathbf{P}$ there is $g \in V$ so that for all $i$ there is $n$ such that $\{ i, g(i) \} \supseteq w_n$. Let $u_n = [g^{(n)}(0), g^{(n+1)}(0))$. By the choice of $I$ there is an infinite $S$ and $\alpha < \omega_1$ so that $\bigcup_{n \in S} u_n \subseteq A_\alpha$. So $S' = \{ n : w_n \subseteq A_n \}$ shows $F$ generates a fat filter.

2.3. Remark. The above proof shows that if $\mu$ is a measure, then $I = \{ A : \mu(A) = 0 \}$ is a quasi-$\sigma$-ring. That is, if $\{ A_n : n < \omega \}$ is a collection of disjoint elements of $I$, then there is an infinite $S$ so that $\bigcup_{n \in S} A_n \in I$.

References


