

PRODUCTS OF CLOSED IMAGES OF CW-COMPLEXES AND k -SPACES

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ABSTRACT. We give some characterizations for the products of closed images of CW-complexes to be k -spaces.

1. Introduction. C. H. Dowker [2] showed that not every product of two CW-complexes is a CW-complex. Liu Ying-ming [6], assuming the continuum hypothesis (CH), gave a necessary and sufficient condition for the product of two CW-complexes to be a CW-complex. The authors (independently, [13 and 16]) showed that Liu's characterization is equivalent to a certain set-theoretic axiom weaker than (CH); see Corollary 3.2(1) in §3.

On the other hand, assuming (CH), the first author [12] gave a characterization for the product of two closed images of metric spaces to be a k -space. G. Gruenhage [4] showed that this characterization is equivalent to an axiom weaker than (CH).

In this paper, we shall give some analogous characterizations for the products of closed images of CW-complexes (or, closed images of spaces dominated by covers of connected, compact metric spaces) to be k -spaces.

Recall that a space X is k -space if it has the *weak topology* with respect to the cover of all compact subsets X_α ; that is, a subset F of X is closed in X whenever $F \cap X_\alpha$ is closed in X_α for every α . Every CW-complex is a k -space.

We assume that all spaces are regular and all maps are continuous and onto.

2. Lemmas. Let us begin with some basic properties of k -spaces; for example, see [3]. We shall omit the proofs.

- LEMMA 2.1. (1) *Every closed or open subset of a k -space is a k -space.*
(2) *Every locally k -space is a k -space.*
(3) *Every quotient (especially perfect) image of a k -space is a k -space.*

Not every product of two k -spaces is a k -space, but the following sufficient conditions for the products are known. The assertions (1), 2(a) and 2(b) are respectively due to [1, 8 and 16]. (MA) means Martin's Axiom.

LEMMA 2.2. (1) *Every product of a locally compact space and a k -space is a k -space.*

(2) *Let X and Y have the weak topology with respect to covers \mathcal{A} and \mathcal{B} of compact subsets respectively. Then $X \times Y$ is a k -space if (a) $|\mathcal{A} \cup \mathcal{B}| \leq \omega$, or (b) (MA). Each element of \mathcal{A} is metric, $|\mathcal{A}| \leq \omega$ and $|\mathcal{B}| < c = 2^\omega$.*

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For a cardinal number α , let S_α (resp. I_α) be the quotient space obtained from the topological sum of α convergent sequences (resp. α closed unit intervals $[0, 1]$) by identifying all the limit points (resp. all the zero points).

Let ${}^\omega\omega$ be the set of all functions from ω to ω . For $f, g \in {}^\omega\omega$, we define $f \geq g$ if $\{n \in \omega; f(n) < g(n)\}$ is finite. Let $\mathbf{b} = \min\{\gamma; \text{there is an unbounded family } A \subset {}^\omega\omega \text{ with } |A| = \gamma\}$, where “unbounded” means in the sense of \geq (so A is unbounded iff no $f \in {}^\omega\omega$ is \geq every $g \in A$). By $BF(\alpha)$, we mean “ $\mathbf{b} \geq \alpha$ ”. It is well-known that (MA) implies “ $\mathbf{b} = c$ ”.

As for the k -ness of the products of the spaces S_α , G. Gruenhagen [4] showed the following result ((a) \Leftrightarrow (c) of (2) is stated in [13 or 16]). α^+ denotes the least cardinal greater than α .

LEMMA 2.3. (1) $S_{\omega_1} \times S_{\omega_1}$ is not a k -space.

(2) The following are equivalent: (a) $BF(\alpha^+)$ holds, (b) $S_\omega \times S_\alpha$ is a k -space, (c) $I_\omega \times I_\alpha$ is a CW-complex.

For a cardinal number α , a space X is k_α if it has the weak topology with respect to a cover \mathcal{C} of compact subsets with $|\mathcal{C}| \leq \alpha$. Let us call a space X locally $< k_\beta$, if each point $x \in X$ has a neighborhood whose closure is k_{α_x} , where $\alpha_x < \beta$. Clearly, every locally $< k_\omega$ -space (resp. locally $< k_{\omega_1}$ -space) is precisely locally compact (resp. locally k_ω).

Let X be a space and $\{X_\alpha\}$ be a closed cover of X . Recall that X is dominated¹ by $\{X_\alpha\}$ if the union of any subcollection $\{X_\beta\}$ of $\{X_\alpha\}$ is closed in X and the union has the weak topology with respect to $\{X_\beta\}$. Every CW-complex is dominated by the closed cover of all finite subcomplexes.

Recall that a space is Fréchet if, whenever $x \in \bar{A}$ then there is a sequence in A converging to the point x .

Let us call a space X a pre- S_α -space, if X admits a perfect map onto the quotient space S_α .

LEMMA 2.4. Suppose that X is dominated by a closed cover of connected, Fréchet and k_ω -spaces X_α . Let β be a regular cardinal. If X contains no closed pre- S_β -subspace, then X is locally $< k_\beta$.

PROOF. Suppose that $\beta \geq \omega_1$ and X is not locally $< k_\beta$ at $x \in X$. Let $\mathcal{F} = \{X_\alpha; x \in X_\alpha\}$, and let $N = \bigcup \mathcal{F}$. Since the union of any subcollection of $\{X_\alpha\}$ is closed in X , $\text{int } N$ is a neighborhood of x . But, N is dominated by \mathcal{F} and each element of \mathcal{F} is k_ω . Since, for each $\gamma < \beta$, N is not k_γ , the set N is not the union of γ many elements of \mathcal{F} . Hence, we can choose a subcollection $\{X_\gamma; \gamma < \beta\}$ of \mathcal{F} such that $X_\gamma - X(\gamma) \neq \emptyset$, where $X(\gamma) = \bigcup_{\delta < \gamma} X_\delta$. But, for each $\gamma < \beta$, $X_\gamma \cup X(\gamma)$ is connected and $X(\gamma)$ is closed in X . So, $\overline{X_\gamma - X(\gamma)} \cap X(\gamma) \neq \emptyset$. Since each X_γ is Fréchet and closed, there exist subsets $\{x_\gamma; 1 \leq \gamma < \beta\}$ and $\{x_{\gamma n}; n \in \omega\}$ of X such that $x_\gamma \in X(\gamma)$, $x_{\gamma n} \in X_\gamma - X(\gamma)$ and $x_{\gamma n} \rightarrow x_\gamma$. Then there is $f: \beta \rightarrow \beta$ such that $f(0) = 0$, $f(\gamma) < \gamma$ with $x_\gamma \in X_{f(\gamma)}$ for $\gamma > 0$. Now, since $\beta \geq \omega_1$ and β is regular, by the Pressing-Down Lemma (for example, see [5, p. 80]) there is a subset $S \subset \beta$ with cardinality β such that $f(S) = \beta_0$ for some $\beta_0 < \beta$. Note that $x_\gamma \in X_{\beta_0}$ for $\gamma \in S$. Since X_{β_0} is σ -compact, some compact subset K of K_{β_0}

¹For a cover \mathcal{C} of a space X , sometimes “ X is dominated by \mathcal{C} ” means the same as “ X has the weak topology with respect to \mathcal{C} ”. But these notions are distinct in this paper.

contains a subset $\{x_{\beta(\gamma)}; \gamma < \beta\}$ with cardinality $\leq \beta$, where $\{\beta(\gamma); \gamma < \beta\} \subset S$ and $\beta(\gamma') < \beta(\gamma)$ if $\gamma' < \gamma$. Let $\mathcal{C} = \{K\} \cup \{L_\gamma; \gamma < \beta\}$ and $X^* = \bigcup \mathcal{C}$, where $L_\gamma = \{x_{\beta(\gamma)}\} \cup \{x_{\beta(\gamma)n}; n \in \omega\}$. Then X^* is a closed subset of X having the weak topology with respect to the cover \mathcal{C} . Indeed, for $F \subset X^*$, let $K' = K \cap F$ and each $L'_\gamma = L_\gamma \cap F$ be relatively closed (hence, K' and each L'_γ are compact in X). Assume that $K' \cup \bigcup_{\gamma < \delta} L'_\gamma$ is closed in X for all $\delta < \alpha$. Let $F_\alpha = K' \cup \bigcup_{\gamma < \alpha} L'_\gamma$. If α is isolated, then F_α is closed in X . So, let α be limit. Let $\mathcal{F}_\alpha = \{X_{\beta_0}\} \cup \{X_{\beta(\gamma)}; \gamma < \alpha\}$. Note that $F_\alpha^* = \bigcup \mathcal{F}_\alpha$ is a closed subset of X having the weak topology with respect to the cover \mathcal{F}_α . But, by the assumption, $F_\alpha \cap X_{\beta_0}$ and each $F_\alpha \cap X_{\beta(\gamma)}$ are closed subsets of F_α^* . Hence, F_α is closed in F_α^* , so is in X . Thus, by induction, $F = F_\beta$ is closed in X (hence, in X^*). This suggests that X^* is a closed subset of X having the weak topology with respect to the cover \mathcal{C} .

Now, let Y be the quotient space obtained from X^* by identifying all the points of the compact set K . Then Y is the space S_β which is a perfect image of X^* . Hence, X contains a closed pre- S_β -subspace X^* . This is a contradiction. Hence, in case $\beta \geq \omega_1$, X is locally $< k_\beta$. In case $\beta = \omega$, X contains no closed pre- S_ω -subspace. So, X contains no closed pre- S_{ω_1} -subspace. Thus, by the above argument, each $x \in X$ has a neighborhood whose closure N_x is dominated by a countable cover of Fréchet k_ω -subspaces, hence is a k_ω -space with Fréchet “pieces”. We remark that the space S_2 of Arens (see [3, Example 1.6.19]) is a pre- S_ω -space. Thus, N_x contains no closed copy of S_ω or of S_2 . Thus, by the proof of Theorem 2.1 in [14] and by Corollary 1.4 in [14], the k_ω -space N_x with Fréchet “pieces” is locally compact. Hence, X is locally compact, so that X is locally $< k_\omega$. That completes the proof.

Every closed map is quotient. Then the following is easily proved, so we shall omit the proof.

LEMMA 2.5. *Let $f: X \rightarrow Y$ be a closed map. If X is dominated by $\{X_\alpha\}$, then Y is dominated by $\{f(X_\alpha)\}$.*

3. Results. Let K be a CW-complex with cells $\{e\}$, and let $C(e)$ be the smallest finite subcomplex containing e ; that is, $C(e)$ is the intersection of all the subcomplexes of K which contain e . Then the following fact (*) is well known:

(*) Any $C(e)$ is connected (for example, see [7]) and compact metric, also K is a paracompact space [9] dominated by $\{C(e)\}$.

Now we are ready for the main theorem.

THEOREM 3.1. *Let X, Y be closed images of CW-complexes. (1) The following are equivalent: (a) $BF(\omega_2)$ is false, (b) $X \times Y$ is a k -space if and only if X or Y is locally compact, otherwise X and Y are locally k_ω (equivalently, locally separable).*

(2) (MA). *$X \times Y$ is a k -space if and only if X or Y is locally compact, otherwise one of X and Y is locally k_ω and another is $< k_c$. When $X = Y$, we can omit (MA).*

PROOF. (1) By Lemmas 2.1 and 2.2, the “if” part of (b) holds. (b) \Rightarrow (a) follows from Lemma 2.3(2). So we will prove the “only if” part of (b) from (a). By the above (*) and Lemma 2.5, X and Y are dominated by covers of connected, compact metric subspaces. Now suppose that neither X nor Y is locally compact. Then, by Lemma 2.4, X and Y contain closed pre- S_ω -subspaces. But, since $X \times Y$

is a k -space, by Lemmas 2.1 and 2.3(2), neither X nor Y contains a closed pre- S_{ω_1} -subspace. Thus, by Lemma 2.4, X and Y are locally k_ω . Since X and Y are dominated by covers of compact metric subspaces, any closed separable (resp. compact) subset of X and Y is contained in a countable (resp. finite) union of these compact metric subspaces, hence is k_ω (resp. separable metric). Thus X and Y are locally k_ω -spaces iff they are locally separable.

(2) By Lemma 2.3, neither $S_{\omega_1} \times S_{\omega_1}$ nor $S_\omega \times S_c$ is a k -space. Thus, similarly we have the “only if” part. The “if” part follows from Lemmas 2.1 and 2.2.

Let us call a CW-complex K *locally* $< \beta$, β is a cardinal, if each $x \in K$ has a neighborhood which meets only α_x many closed cells \bar{e} , where $\alpha_x < \beta$.

Now, as is well-known, any compact subset of a CW-complex meets only finitely many cells. Also, every product of two CW-complexes is a CW-complex if and only if it is a k -space [11, Lemma 4.4]. Thus, by Theorem 3.1, we have the following result in [16] (for (1), also see [13]).

COROLLARY 3.2. *Let K, L be CW-complexes.*

(1) *The following are equivalent: (a) $BF(\omega_2)$ is false, (b) $K \times L$ is a CW-complex (equivalently, a k -space) if and only if K or L is locally finite, otherwise K and L are locally countable.*

(2) *$K \times L$ is a CW-complex if and only if K or L is locally finite, otherwise one of K and L is locally countable and another is locally $< c$. When $K = L$, we can omit (MA).*

COROLLARY 3.3. *Let Y be the closed image of a CW-complex.*

(1) *Y^2 is a k -space if and only if Y is a locally k_ω -space (equivalently, a locally separable space).*

(2) *Y^ω is a k -space if and only if Y is a locally compact metric space.*

PROOF. Since (1) follows from the latter part of Theorem 3.1(2), we will prove (2). Let Y^ω be a k -space. Since $(S_\omega)^\omega$ is not a k -space by [10, Theorem 1.3], Y contains no closed pre- S_ω -subspace by Lemma 2.1. Thus, by Lemma 2.4, Y is locally compact. Hence the paracompact space Y is locally metric, so Y is metric.

REMARK 3.4. Theorem 3.1 and Corollary 3.3 hold if we replace “CW-complex(es)” by “space(s) dominated by connected, locally compact and separable metric subspaces”. Indeed, such subspaces are k_ω . Thus, using Lemmas 2.4 and 2.5, this replacement is possible.

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