ON A THEOREM OF INGHAM
ON NONHARMONIC FOURIER SERIES
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ABSTRACT. A well-known result due to Ingham [3] shows that the system of complex exponentials \( \{e^{i\lambda_n t}\} \) is a basic sequence in \( L^2(-\pi, \pi) \) whenever \( \lambda_{n+1} - \lambda_n \geq \gamma > 1 \). In this note, we show that the system need not be basic if \( \lambda_{n+1} - \lambda_n > 1 \).

1. Introduction. Let \( \{\lambda_n\} \) be an increasing sequence of real numbers. A well-known result due to Ingham [3] states that if \( \lambda_{n+1} - \lambda_n \geq \gamma > 1 \) then the series \( \sum c_n e^{i\lambda_n t} \) converges in \( L^2(-\pi, \pi) \) whenever \( \sum |c_n|^2 < \infty \) and, moreover,

\[
A \sum |c_n|^2 \leq \left( \sum |c_n e^{i\lambda_n t}|^2 \right) \leq B \sum |c_n|^2.
\]

Here, \( A \) and \( B \) are positive constants depending only on \( \gamma \). (That the right-hand inequality is valid whenever \( \gamma > 0 \) appears to have been first proved by Titchmarsh [10].)

Ingham showed that his result is the best possible in the sense that if \( \gamma = 1 \) then the left-hand inequality cannot obtain. A counterexample is provided by the sequence \( \{\lambda_n\} \) where

\[
\lambda_n = \begin{cases} 
n - 1/4, & n > 0, \\
n + 1/4, & n < 0. 
\end{cases}
\]

It follows readily from (1) that the system of exponentials \( \{e^{i\lambda_n t}\} \) is a basic sequence in \( L^2(-\pi, \pi) \), that is, a basis for its closed linear span \( S \). Accordingly, each function \( f \) in \( S \) has a unique representation

\[
f(t) = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean)}.
\]

The study of such nonharmonic Fourier series was initiated by Paley and Wiener [5] who showed that the system \( \{e^{i\lambda_n t}\} \) is a basis for \( L^2(-\pi, \pi) \) whenever the \( \lambda_n \) are sufficiently close to the integers. Since then the theory has been generalized in many ways and in many different directions (see, e.g., [2, 6, 8, 11] and the references therein).

Condition (1), while tractable, is a stringent requirement to place on a basic sequence. Nevertheless, we show in Theorem 1 that the right-hand inequality must obtain for every basic sequence of exponentials. At present, there is no known example of such a sequence for which the left-hand inequality does not also obtain. Theorem 2 further dramatizes the strength of Ingham’s result by showing that the slightly weaker separation condition \( \lambda_{n+1} - \lambda_n > 1 \) cannot even guarantee that the system \( \{e^{i\lambda_n t}\} \) is a basic sequence in \( L^2(-\pi, \pi) \).
THEOREM 1. If \( \{\lambda_n\} \) is an increasing sequence of real numbers for which the system of exponentials \( \{e^{i\lambda_n t}\} \) is a basic sequence in \( L^2(-\pi, \pi) \), then the inequality \( \| \sum c_n e^{i\lambda_n t} \|^2 \leq B \sum |c_n|^2 \) is valid for some constant \( B \) and all square summable sequences of scalars \( \{c_n\} \).

THEOREM 2. There exists a sequence \( \{\mu_n\} \) of real numbers satisfying \( \mu_{n+1} - \mu_n > 1 \) such that \( \{e^{i\mu_n t}\} \) is exact in \( L^2(-\pi, \pi) \) and yet not a basis.

Recall that \( \{e^{i\mu_n t}\} \) is said to be exact if it is complete but fails to be complete upon the removal of a single term.

2. Proof of Theorem 1. We need only show that the \( \lambda_n \) are separated, i.e., that \( \lambda_{n+1} - \lambda_n \geq \gamma \) for some positive constant \( \gamma \); the result will then follow from [10].

Let \( S \) be the closure in \( L^2(-\pi, \pi) \) of the linear span of the system \( \{e^{i\lambda_n t}\} \), and let \( \{f_n\}, f_n \in S^* \), be the associated sequence of coefficient functionals. Then \( \|e^{i\lambda_n t}\| \|f_n\| \leq M \) for some constant \( M \) and all values of \( n \) (see, e.g., [9, p. 20]). Since each \( \lambda_n \) is real, \( \|e^{i\lambda_n t}\| = 1 \) and hence \( \|f_n\| \leq M \). Now \( f_n(e^{i\lambda_n t} - e^{i\lambda_{n+1} t}) = 1 \) so that \( \|f_n\| \|e^{i\lambda_n t} - e^{i\lambda_{n+1} t}\| \geq 1 \). Accordingly, \( \|e^{i\lambda_n t} - e^{i\lambda_{n+1} t}\| \geq 1/M \) and the existence of \( \gamma \) follows.

3. Proof of Theorem 2. The system \( \{e^{i\lambda_n t}\} \) where the \( \lambda_n \) are given by (1), is known to be exact in \( L^2(-\pi, \pi) \) [4, p. 67]. We begin by showing that it is not a basis. Suppose it were. Then we could write

\[
1 = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean).}
\]

To compute the \( c_n \), we shall make use of the Paley-Wiener space \( P \) consisting of all entire functions of exponential type at most \( \pi \) that are square integrable on the real axis. The inner product of two functions \( F \) and \( G \) in \( P \) is, by definition,

\[
(F, G) = \int_{-\infty}^{\infty} F(x)G(x) dx.
\]

By virtue of the Paley-Wiener theorem, the complex Fourier transform

\[
f(t) \to \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{ist} dt
\]

is an isometric isomorphism from \( L^2(-\pi, \pi) \) onto all of \( P \). The exponentials \( e^{i\lambda_n t} \) are sent to the "reproducing" functions

\[
K_n(z) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)}
\]

which then consistute a basis for \( P \). Let \( \{g_n\} \) be biorthogonal to \( \{K_n\} \). When the Fourier transform is applied to (3), we obtain

\[
\frac{\sin \pi z}{\pi z} = \sum c_n K_n(z)
\]

where \( c_n = ((\sin \pi z)/\pi z, g_n) = g_n(0) \).

Let

\[
F(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{\lambda_n^2} \right).
\]
It was shown by Levinson [4, p. 67] that

\[ F(z) = c \int_{-\pi}^{\pi} \left( \cos^{-1/2} \frac{1}{2} t \right) e^{izt} dt. \]

Since \( \cos^{-1/2} \frac{1}{2} t \) is integrable over \((-\pi, \pi)\), it follows that \( F(z) \) is bounded along the real axis and each of the functions

\[ F_n(z) = F(z)/F'(\lambda_n)(z - \lambda_n) \]

therefore belongs to \( P \). Since \( (F_n, K_m) = F_n(\lambda_m) = \delta_{mn} \), it follows that \( F_n = g_n \). Thus

\[ c_n = F_n(0) = -1/\lambda_n F'(\lambda_n), \]

and (3) becomes

\[ (4) \quad 1 = -\sum_{n \neq 0} \frac{e^{i\lambda_n t}}{\lambda_n F'(\lambda_n)} = -2 \sum_{n=1}^{\infty} \frac{\cos \lambda_n t}{\lambda_n F'(\lambda_n)} \]

since \( zF'(z) \) is even. It is to be shown that the series on the right does not converge in \( L^2(-\pi, \pi) \).

Now the values of \( F'(\lambda_n) \) were determined explicitly in [7]:

\[ F'(\lambda_n) = (-1)^n \Gamma^2 \left( \frac{3}{4} \right) \frac{\Gamma(n)}{\Gamma(n + \frac{1}{2})} \quad (n = 1, 2, 3, \ldots). \]

Using the asymptotic formula \( \Gamma(n)/\Gamma(n + \frac{1}{2}) = 1/\sqrt{n} + O(1/n^{3/2}) \) [1], we have

\[ F'(\lambda_n) = A(-1)^n \left\{ 1/\sqrt{\lambda_n} + \varepsilon_n \right\} \quad \text{where} \quad \varepsilon_n = O(1/n^{3/2}). \]

A straightforward calculation then shows that the difference between the series on the right in (4) and

\[ \frac{1}{A} \sum_{n=1}^{\infty} \frac{(-1)^n \cos \lambda_n t}{\sqrt{\lambda_n}} \]

is uniformly convergent on \([-\pi, \pi]\). Accordingly, we need only show that this series diverges in \( L^2(-\pi, \pi) \).

Let \( x = \pi - t \) \((0 \leq t \leq \pi)\). Then

\[ \sum_{n=1}^{\infty} \frac{(-1)^n \cos \lambda_n t}{\sqrt{\lambda_n}} = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\pi n - \pi/4 - \lambda_n x)}{\sqrt{\lambda_n}} \]

\[ = \sum_{n=1}^{\infty} \frac{\cos(\pi/4 + \lambda_n x)}{\sqrt{\lambda_n}}. \]

For \( N = 1, 2, 3, \ldots \), let \( \delta_N = \pi/16N \). If \( x \in [0, \delta_N] \), then \( \pi/4 + \lambda_n x \in [\pi/4, 3\pi/8] \) whenever \( 1 \leq n \leq 2N \) and hence \( \cos(\pi/4 + \lambda_n x) \geq A > 0 \). Thus

\[ \sum_{N}^{2N} \frac{\cos(\pi/4 + \lambda_n x)}{\sqrt{\lambda_n}} \geq A \sum_{N}^{2N} \frac{1}{\sqrt{n - 1/4}} \geq A \frac{N + 1}{\sqrt{2N - 1/4}} \geq B \sqrt{N} \]
where $B$ is a positive constant independent of $N$. Accordingly,

$$
\left\| \sum_{N} \frac{(-1)^{n} \cos \lambda_{n} t}{\sqrt{\lambda_{n}}} \right\|^{2} \geq \left\| \sum_{N} \frac{\cos(\pi/4 + \lambda_{n} x)}{\sqrt{\lambda_{n}}} \right\|^{2}_{L^{2}(0,\delta_{N})}
\geq \frac{1}{2\pi} B^{2} N \delta_{N} = \frac{B^{2}}{32}
$$

for all $N$. Thus the series in (3) does not converge in $L^{2}(-\pi, \pi)$, and the system \{e^{i\lambda_{n} t}\} fails to be a basis.

Let \{e, \varepsilon_{2}, \ldots\} be a decreasing sequence of positive numbers such that $\varepsilon_{1} < \frac{1}{4}$ and $\sum \varepsilon_{n} < \infty$. It is to be shown that if $\mu_{n} = \lambda_{n} - \varepsilon_{n}$, $\mu_{n-1} = -\mu_{n}$ ($n = 1, 2, \ldots$), then the system \{e^{i\mu_{n} t}\} satisfies the conclusions of the theorem.

Clearly, $\mu_{n+1} - \mu_{n} > 1$. Since $\sum |\lambda_{n} - \mu_{n}| < \infty$ and \{e^{i\lambda_{n} t}\} is exact, so is \{e^{i\mu_{n} t}\} [6]. It remains only to show that \{e^{i\mu_{n} t}\} is not a basis for $L^{2}(-\pi, \pi)$. Suppose it were. Then the system \{h_{n}(t)\}, biorthogonal to \{e^{i\mu_{n} t}\}, would satisfy $\|e^{i\mu_{n} t}\| \cdot \|h_{n}\| \leq M$ for some constant $M$ and all values of $n$. Since each $\mu_{n}$ is real, $\|e^{i\mu_{n} t}\| = 1$ and hence $\|h_{n}\| \leq M$. We complete the proof by showing that

$$1 \leq \sum |\lambda_{n} - \mu_{n}| < \infty \Rightarrow \sum \|e^{i\lambda_{n} t} - e^{i\mu_{n} t}\| < \infty.$$ 

The convergence of $\sum \|e^{i\lambda_{n} t} - e^{i\mu_{n} t}\| \|h_{n}\|$ will then imply that \{e^{i\lambda_{n} t}\} is a basis for $L^{2}(-\pi, \pi)$ (see, e.g., [9, p. 94]). The contradiction will prove the theorem.

To establish (5), write

$$e^{i\lambda_{n} t} - e^{i\mu_{n} t} = e^{i\lambda_{n} t}(1 - e^{-i\varepsilon_{n} t}).$$

Expanding $1 - e^{i\gamma t}$ in an everywhere-convergent Taylor series, we find

$$|e^{i\lambda_{n} t} - e^{i\mu_{n} t}| \leq \sum_{K=1}^{\infty} \frac{\varepsilon_{n}^{K} K}{K!},$$

and hence

$$\sum_{n} \|e^{i\lambda_{n} t} - e^{i\mu_{n} t}\| \leq \sum_{K=1}^{\infty} \frac{\pi K}{K!} \left(\sum_{n} \varepsilon_{n}^{K}\right) \leq \sum_{K=1}^{\infty} \frac{\pi K}{K!} \left(\sum \varepsilon_{n}\right)^{K} = \exp(\pi \sum \varepsilon_{n}) - 1 < \infty.$$ 

This completes the proof.

REFERENCES


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