ARRANGEMENTS OF LINES
WITH A LARGE NUMBER OF TRIANGLES

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Abstract. An arrangement of lines is constructed by choosing \( n \) diagonals of the regular \( 2n \)-gon. This arrangement is proved to form at least \( n(n - 3)/3 \) triangular cells.

1. Introduction. We shall use the terminology of Grünbaum [5]. By an arrangement \( \mathcal{A} \) of lines we mean a finite family of lines \( L_1, \ldots, L_n \) in the real projective plane \( \mathbb{P}^1 \). The number of lines in \( \mathcal{A} \) is denoted by \( n(\mathcal{A}) \). If no point belongs to more than two of these lines \( L_i \), the arrangement is called simple. With an arrangement \( \mathcal{A} \) there is associated a 2-dimensional cell-complex into which the lines of \( \mathcal{A} \) decompose \( \mathbb{P}^1 \). It is well known that in a simple arrangement \( \mathcal{A} \) the number of cells (or polygons) of that complex is \( (n^2 - n + 2)/2 \) \( (n = n(\mathcal{A})) \). We shall denote by \( p_j(\mathcal{A}) \) the number of \( j \)-gons among the cells of (the complex associated with) \( \mathcal{A} \).

2. Constructions. Let us denote by \( P(O) \) a fixed point on the circle \( \mathcal{C} \) with centre \( C \). For any real \( \alpha \), let \( P(\alpha) \) be the point obtained by rotating \( P(O) \) around \( C \), with angle \( \alpha \). Further denote by \( L(\alpha) \) the straight line \( P(\alpha) P(\pi - 2\alpha) \). In case \( \alpha \equiv \pi - 2\alpha \) \( (\text{mod } 2\pi) \), \( L(\alpha) \) is the line tangent to \( \mathcal{C} \) at \( P(\alpha) \). (See Figure 1.)

![Diagram of Constructions](image_url)
Figure 2

Figure 3
Example 1. Given any integer \( n \geq 3 \), put
\[ \mathcal{A}_n = \{ L((2i + 1)\pi/n) : i = 0, 1, \ldots, n - 1 \}. \]

(See Figure 2.)

Example 2. Given any integer \( n \geq 3 \), put
\[ \mathcal{B}_n = \{ L(2i\pi/n) : i = 0, 1, \ldots, n - 1 \}. \]

(See Figure 3.)

Remark that our set of lines \( \{ L(\alpha) : 0 \leq \alpha < 2\pi \} \) may be regarded as a set of tangents to the arcs of a hypocycloid of third order, drawn in a circle of centre \( C \) and radius 3. The line \( L(\alpha) \) is tangent to the arc of the cycloid at the \( \alpha \)th point. (See Figure 4.) However, we shall not rely upon this fact in what follows.

Lemma. The lines \( L(\alpha), L(\beta) \) and \( L(\gamma) \) are concurrent if and only if \( \alpha + \beta + \gamma \equiv 0 \pmod{2\pi} \).

Proof. If \( \alpha + \beta + \gamma \equiv 0 \pmod{2\pi} \), then the sum of lengths of (directed) arcs \( (P(\alpha), P(\beta)) \) and \( (P(\beta), P(\pi - 2\gamma)) \) is equal to \( \pi \). This implies that \( L(\gamma) \) is perpendicular to the line \( P(\alpha)P(\beta) \). Hence, the lines \( L(\alpha), L(\beta) \) and \( L(\gamma) \) are the altitudes of the triangle \( P(\alpha)P(\beta)P(\gamma) \) (see Figure 1), consequently they meet at one point.

The reverse can be proved similarly.

3. Triangles in a simple arrangement. Grünbaum [5] (cf. Theorem 2.21) pointed out that the maximal number of triangles in a simple arrangement \( p_3(n) = \max\{ p_3(\mathcal{A}) : n(\mathcal{A}) = n, \mathcal{A} \text{ is simple} \} \) can be estimated by \( p_3(n) \leq \frac{n(n - 1)}{3} \), for even \( n \), and \( p_3(n) \leq \frac{n(n - 2)}{3} \), if \( n \) is odd. Moreover, he conjectured that this latter inequality holds for all \( n, n \not\equiv 4 \pmod{6} \). The exact value of \( p_3(n) \) is known only for some small values of \( n \). (Cf., e.g., Simmons [15, 16] for the case \( n = 15 \), Grünbaum [5] for \( n = 20 \) and Harborth [8] for \( n = 17 \).)
As far as we know, the best lower bound by now, $p_3(n) > (5/16 + o(1))n^2$, was due to Füredi and J. Pach (unpublished). We are now in a position to establish a sharper lower bound.

**Property 1.** $p_3(\mathcal{A}_n) \geq n(n - 3)/3$; hence $p_3(n) = n^2/3 + O(n)$.

**Proof.** Let $(L_i, L_j)$ denote the intersection point of lines $L_i = L((2i + 1)\pi/n)$ and $L_j = L((2j + 1)\pi/n)$.

Using the Lemma, we obtain that only $L((2n - 2i - 2j - 2)\pi/n)$ may cross $L_i$ and $L_j$ at $(L_i, L_j)$. But this line does not belong to $\mathcal{A}_n$, by definition. Therefore, the lines $L_i, L_j, L_{n-i-j-1}$ and $L_i, L_j, L_{n-i-j-2}$, resp., necessarily form triangular cells.

4. Quadrangles in a simple arrangement. Grünbaum conjectured that $p_4(\mathcal{A}) \geq 1$ for any simple arrangement with $n(\mathcal{A}) > 16$. (See 2.12 in [5], cf. [15, 16].) As far as we know one cannot find in the literature any example of a simple arrangement containing $\leq o(n^2)$ quadrangles. In view of this, Grünbaum’s conjecture is surprisingly modest. It is easy to prove that

**Property 2.** The simple arrangement $\mathcal{A}_n$ contains only 3 or 5 quadrangles for odd $n$, and $p_4(\mathcal{A}_n) = O(n)$ is valid for all values of $n$.

The results of §§3 and 4 are collected in Table 1. (We remark that $p_j(\mathcal{A}_n) = 0$ for $j \geq 7$.)

<table>
<thead>
<tr>
<th>$n \geq 5$</th>
<th>$n \equiv 0 \pmod{6}$</th>
<th>$n \equiv 1 \pmod{6}$</th>
<th>$n \equiv 2 \pmod{6}$</th>
<th>$n \equiv 3 \pmod{6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_3(\mathcal{A}_n)$</td>
<td>$\frac{1}{3}(n^2 - 3n)$</td>
<td>$n/2 + 6$</td>
<td>$n - 6$</td>
<td>$\frac{1}{6}(n^2 - 6n + 6)$</td>
</tr>
<tr>
<td>$p_4(\mathcal{A}_n)$</td>
<td>$\frac{1}{3}(n^2 - 3n + 5)$</td>
<td>5</td>
<td>$2n - 9$</td>
<td>$\frac{1}{6}(n^2 - 9n + 20)$</td>
</tr>
<tr>
<td>$p_5(\mathcal{A}_n)$</td>
<td>$\frac{1}{3}(n^2 - 3n + 8)$</td>
<td>$n/2$</td>
<td>$n - 2$</td>
<td>$\frac{1}{6}(n^2 - 6n + 2)$</td>
</tr>
<tr>
<td>$p_6(\mathcal{A}_n)$</td>
<td>$\frac{1}{3}(n^2 - 3n + 9)$</td>
<td>3</td>
<td>$2n - 9$</td>
<td>$\frac{1}{6}(n^2 - 9n + 24)$</td>
</tr>
</tbody>
</table>

5. Triangles in arbitrary arrangements. Grünbaum conjectures that for any arrangement $\mathcal{A}$ of $n$ lines, $p_3(\mathcal{A}) \leq n(n - 1)/3$ holds. (See [5].) Let $p_3(n) = \max\{p_3(\mathcal{A}_n): n(\mathcal{A}) = n\}$. The best upper bound was given by Purdy [11, 12], who proved that

$$p_3(n) \leq \frac{7}{18}n(n - 1) + \frac{1}{3} \text{ for } n \geq 6.$$

The best lower bound, $p_3(n) \geq 4 + n(n - 3)/3$, is due to Strommer [18]. His result uses a construction of Burr, Grünbaum and Sloane [1].

**Property 3.** $p_3(\mathcal{B}_n) \geq 4 + n(n - 3)/3$.

More exactly $p_3(\mathcal{B}_n) = n(n - 3)/3 + 6 - 2\varepsilon/3$, where $\varepsilon = 0, 2, 2$ according to whether $n \equiv 0, 1, 2 \pmod{3}$. Further, $p_4(\mathcal{B}_n) = n - 6 + \varepsilon$, and $p_j(\mathcal{B}_n) = 0$ for $j \geq 5$. The proof is easy.

6. The orchard problem. Given a vertex $V$ in an arrangement $\mathcal{A}$, denote by $t(V, \mathcal{A})$ the multiplicity of $V$, i.e. the number of lines of $\mathcal{A}$ incident to $V$. Further, let $t_j(\mathcal{A})$ denote the number of vertices of multiplicity $j$ ($2 \leq j \leq n$). We use the
notation $t_j(n) = \max\{t_j(\mathcal{A}) : n(\mathcal{A}) = n\}$. The "orchard problem" has been investigated for about 150 years. It can be formulated as follows: find the value of $t_3(n)$. Significant progress has been made by Burr, Grünbaum and Sloane [1]. They proved that $t_3(n) \geq 1 + \frac{n(n - 3)}{6}$ by construction using elliptic integrals. Moreover, they conjectured that this result is sharp if $n$ is large enough. (Confer [1] for a complete (historical) bibliography of the subject.) Our construction (see Example 2) is much simpler, but this is only a special case of their idea. (The equation of the poles of $L(a)$ is $(x^2 + y^2)(3x - 1) = 4x^3$. This can be transformed into the form $y^2 = 4x^3 - (1/12)x - (1/216)$. Cf. [1].)

**Property 4.** $t_2(\mathcal{R}_n) = n - 3 + \varepsilon$, $t_3(\mathcal{R}_n) = 1 + \frac{n(n - 3)}{6}$, and $t_j(\mathcal{R}_n) = 0$ for $j \geq 4$.

It should be noted that recently Szemerédi and Trotter [19] proved that there exist $c$ and $c'$ positive real numbers such that $cn^2/k^3 < m(t_j(\mathcal{A})) < c'n^2/k^3$ for all $n > k^2$.

7. **Two-coloring of arrangements.** It is easy to prove by induction that the cells of a (not necessarily simple) arrangement $\mathcal{A}$ in the Euclidean plane can be colored by two colors (e.g., black and white) so that any two regions with a common side get different colors. Let $b = b(\mathcal{A})$ and $w = w(\mathcal{A})$ denote the numbers of black and white polygons. Without loss of generality we can assume that $b \geq w$. L. Fejes Tóth [3] raised the following question: What is the maximum of the ratio $b/w$? Palásti [10] proved that $b/w < 2$ for $n(\mathcal{A}) \leq 9$. Upper bounds were given by Grünbaum [6], Simmons and Wetzel [17] and for higher dimensions by Purdy and Wetzel [13]. However, the exact value of max $b/w$ is known only for some small values of $n$ with $n \leq 16$.

Grünbaum proved that $b \leq 2w - 2$ for all arrangements $\mathcal{A}$ with $n(\mathcal{A}) \geq 3$. For $n = 3, 5, 9$ and 15 equality holds. Our Example 1 shows

**Property 5.** The number of black regions $b(\mathcal{A}_n) = (n^2 + \varepsilon)/3$, where $\varepsilon = 0, 2, 2$ if $n \equiv 0, 1, 2 \pmod{3}$. So we get $b(\mathcal{A}_n) = 2w(\mathcal{A}_n) - (n + 2 - \varepsilon)$.

8. **Gallai points.** Let $\mathcal{A}$ be an arrangement of $n$ lines of the projective plane such that it does not contain a common point (i.e., $t_n(\mathcal{A}) = 0$). T. Gallai [4] proved that in this case there exist two lines from $\mathcal{A}$ whose intersection point has multiplicity 2. This statement was improved by Kelly and Moser [9] ($t_2(\mathcal{A}) \geq 3n/7$) and recently by S. Hansen [7] ($t_2(\mathcal{A}) \geq n/2$).

The following question was posed by P. Erdős [2]. Let us suppose that the arrangement $\mathcal{A}$ does not contain a point with multiplicity more than 3. Then does there exist a Gallai triangle, i.e. three lines from $\mathcal{A}$ such that their three intersection points have multiplicity 2, or not? Our construction $\mathcal{B}_n$ shows that the answer is negative for $n \geq 4$, $n \not\equiv 0 \pmod{9}$. Another problem on Gallai points can be found in [2].

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References