NORMAL SUBGROUPS OF $\text{Diff}^\Omega(\mathbb{R}^3)$

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ABSTRACT. Let $\Omega$ be a volume element on $\mathbb{R}^3$ of infinite total $\Omega$-volume. We denote by $\text{Diff}^\Omega(\mathbb{R}^3)$ the group of all $\Omega$-preserving diffeomorphisms of $\mathbb{R}^3$, by $\text{Diff}_c^\Omega(\mathbb{R}^3)$ the subgroup of all elements with compact support and by $\text{Diff}_f^\Omega(\mathbb{R}^3)$ the subgroup of all elements whose support has finite $\Omega$-volume.

We prove that there is no normal subgroup between $\text{Diff}_c^\Omega(\mathbb{R}^3)$ and $\text{Diff}_f^\Omega(\mathbb{R}^3)$.

In my paper Normal subgroups of $\text{Diff}^\Omega(\mathbb{R}^n)$ [5], I studied the normal subgroups of $\text{Diff}^\Omega(\mathbb{R}^n)$ for $n \geq 4$, $\Omega$ being any volume element on $\mathbb{R}^n$. All results in the paper hold for $n = 3$ except Lemma 4.4. Thus we know that the normal subgroup of $\text{Diff}^\Omega(\mathbb{R}^3)$ of all elements compactly $\Omega$-isotopic to the identity, $\text{Diff}_c^\Omega(\mathbb{R}^3)$, is simple, and there is a maximal proper normal subgroup of $\text{Diff}^\Omega(\mathbb{R}^3)$, $\text{Diff}_f^\Omega(\mathbb{R}^3)$, the subgroup of all elements with set of nonfixed points of finite $\Omega$-volume.

The purpose of this paper is to prove a modification of Lemma 4.4 of [5] for $n = 3$, getting, as a consequence, that there is no normal subgroup between $\text{Diff}_c^\Omega(\mathbb{R}^3)$ and $\text{Diff}_f^\Omega(\mathbb{R}^3)$.

The importance of Lemma 4.4 is given by the fact that the basic method for understanding the normal subgroups is to factor a diffeomorphism into a product of diffeomorphisms whose support is well-controlled and then to manipulate this support using techniques of [2].

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Let us start by giving some definitions on infinite links.

DEFINITION. Let $\bigcup_{i \geq 1} \alpha_i, \bigcup_{i \geq 1} \beta_i$ be two locally finite sets of disjoint smooth paths in $\mathbb{R}^3$ such that $\alpha_i \cap \beta_j = \emptyset$ if $i \neq j$ and

$$\alpha_i \cap \beta_i = (\alpha_i(0) = \beta_i(0)) \cup (\alpha_i(1) = \beta_i(1)).$$

Let $p: \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ given by $p(x,y,z) = (x,y,0)$ be the parallel projection.

We call a crossing of the link $L = \bigcup_{i \geq 1} \alpha_i \cup \bigcup_{i \geq 1} \beta_i$ the set of points $p^{-1}(c)$, where $c$ is a multiple point of $p|_L$. When no confusion is possible we also call the point $c$ a crossing.

Since every differentiable knot is equivalent to one in regular position, and since in $L$ we have a locally finite sequence of differentiable paths, we can assume that all crossings are double. Let $c$ be a double point of $p|_L$. We call $c'$ the point of $p^{-1}(c)$ with larger $z$-coordinate and $c''$ the other one.

Now, we have two different types of crossings:

(a) $p^{-1}(c) \subset \alpha_i \cup \alpha_j$ or $p^{-1}(c) \subset \beta_i \cup \beta_j$. 

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DEFINITION. A crossing $p^{-1}(c)$ is an overcrossing and we denote it by “O” in the following cases:

(i) Type (a): if $c'$ lies in $\alpha_i$ when $i < j$ or if we first find $c'$ when $\alpha_i$ is traversed from $\alpha_i(0)$ to $\alpha_i(1)$ if $i = j$; similarly if $p^{-1}(c) \subset \beta_i \cup \beta_j$.

(ii) Type (b): if $c'$ lies in $\alpha_i$ when $i \leq j$ or in $\beta_j$ when $j < i$.

Otherwise, we call a crossing an undercrossing and we denote it by “U”.

We now prove

**Lemma.** Let $L$ be as above. There are smooth paths $\bigsqcup_{i \geq 1} \alpha'_i$, $\bigsqcup_{i \geq 1} \beta'_i$ such that $\alpha'_i$ is very near $\alpha_i$ and $\beta'_i$ is very near $\beta_i$, $\alpha'_i \cap \beta'_j = \emptyset$ if $i \neq j$, $\alpha'_i \cap \beta'_i = (\alpha'_i(0) = \beta'_i(0)) \cup (\alpha'_i(1) = \beta'_i(1))$, and all crossings of $(\bigsqcup_{i \geq 1} \alpha'_i) \cup (\bigsqcup_{i \geq 1} \beta'_i)$ are overcrossings.

**Proof.** We define $\alpha'_i, \beta'_i$ inductively on $i$.

$\alpha'_i, \beta'_i$ are different from $\alpha_i, \beta_i$ only in a chosen neighbourhood of each undercrossing $U = p^{-1}(c)$ where $\alpha'_i$ and $\beta'_i$ are defined as follows.
(I) $U$ is of type (a). On a neighbourhood of $c'$, $\alpha'_j$ (resp. $\beta'_j$) goes vertically (in the $z$-direction) under $\alpha_i$ (resp. $\beta_i$) instead of over. On a neighbourhood of $c''$, $\alpha'_i$ (resp. $\beta'_i$) is the same as $\alpha_i$ (resp. $\beta_i$) (see Figure 2).

(II) $U$ is of type (b). $\alpha'_i$ is $\alpha_i$. On a neighbourhood of $c'$, $\beta'_i$ goes vertically (in the $z$-direction) under $\alpha_i$ instead of over it if $i \leq j$; if $i > j$, on a neighbourhood of $c''$, $\beta'_i$ goes vertically (also in the $z$-direction) over $\alpha_i$ instead of under.

Thus, all crossings of $\bigsqcup_{i \geq 1} \alpha'_i \cup \bigsqcup_{i \geq 1} \beta'_i$ are overcrossings.

**Remark.** We know by McDuff [6] that the loops $\alpha_i \cup \alpha'_i$ and $\beta_i \cup \beta'_i$ are both unknotted for any $i$.

Furthermore, notice that the infinite link $\bigsqcup_{i \geq 1} \alpha'_i \cup \bigsqcup_{i \geq 1} \beta'_i$ constructed above is untangled in the sense that it is diffeomorphic to the standard one.

**Figure 4**
Before proving Lemma 4.4 for \( n = 3 \) we will define a strip.

**DEFINITION.** A strip in \( \mathbb{R}^3 \) is the image under some diffeomorphism of \( \mathbb{R}^3 \), \( g \), of the tube \( \{(x, y, z) \in \mathbb{R}^3: x > 0, y^2 + z^2 \leq 1\} \).

Notice that a strip may have finite \( \Omega \)-volume since \( g \) may not be volume preserving.

We now state and prove Lemma 4.4 for \( n = 3 \).

**THEOREM.** Let \( f \) be any volume element of \( \text{Diff}^\Omega(\mathbb{R}^3) \) with support in a strip \( V \) of infinite \( \Omega \)-volume. Then \( f = f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 \circ f_6 \), where \( f_i \) lies in \( \text{Diff}^\Omega(\mathbb{R}^3) \) and has support in a strip \( V_i \) of finite \( \Omega \)-volume.

**PROOF.** As in Lemma 4.4 of [5] we get a disjoint union of closed balls \( \bigsqcup_{i \geq 1} B_i \subset \text{int} V - \text{supp} f \) such that \( \text{vol}_\Omega(V - \bigsqcup_{i \geq 1} B_i) < \infty \). We can join each ball \( B_i \) to \( \partial V \) by an unknotted smooth path \( \alpha_i \) in \( V \) satisfying:

(i) The set \( \{\alpha_i\} \) is locally finite.

(ii) \( \alpha_i \cap \alpha_j = \emptyset \) if \( i \neq j \).

(iii) \( \alpha_i \cap \beta_j = \emptyset \) if \( i \neq j \) and \( \alpha_i \cap B_i = \alpha_i(1) \). Also, we can get \( f_1 \), a volume preserving diffeomorphism with support in a strip of finite \( \Omega \)-volume such that \( f_1^{-1} \circ f(\alpha_i) \cap \alpha_j = \emptyset \) for any \( i \neq j \) and \( f_1^{-1} \circ f(\alpha_i) \) and \( \alpha_i \) only meet on a connected neighbourhood of its endpoints.

We consider now the infinite link \( L = \bigsqcup_{i \geq 1} \alpha_i \cup \bigsqcup_{i \geq 1} \beta_i \), where \( \beta_i = f_1^{-1} \circ f(\alpha_i) \), and we apply the Lemma to it. So we get, for any \( i \), \( \alpha'_i = \alpha_i \) because the \( \alpha_i \) never cross each other and \( \bigsqcup_{i \geq 1} \beta'_i \), where \( \beta'_i \) is different from \( f_1^{-1} \circ f(\alpha_i) \) only in a small neighbourhood of each undercrossing. \( \bigsqcup_{i \geq 1} \alpha_i \cup \bigsqcup_{i \geq 1} \beta_i' \) is untangled and for any \( i \), \( \alpha_i \cup \beta'_i \) and \( \beta_i \cup \beta'_i \) are both unknotted.

Let \( \beta''_i \) be the path \( f_1^{-1} \circ f(\alpha_i) \) except near an undercrossing of type (b) where we have changed it to an overcrossing as in the Lemma. So there is a volume preserving diffeomorphism, \( f_2 \), with support in a disjoint union of cells of \( \Omega \)-volume as small as we like such that \( f_2^{-1}(\beta_i) = f_2^{-1} \circ f_1^{-1} \circ f(\alpha_i) = \beta''_i \).
Now we consider the link $\bigcup_{i \geq 1} \beta'_i \cup \bigcup_{i \geq 1} \beta''_i$. In the same way as in [6, Lemma 8], we can prove that the link $\bigcup_{i \geq 1} \beta'_i \cup \bigcup_{i \geq 1} \beta''_i$ is untangled, therefore, there is a volume preserving diffeomorphism, $f_3^{-1}$, with support in a disjoint union of cells of $\Omega$-volume as small as we like such that $f_3^{-1}(\beta''_i) = \beta'_i$ for any $i$.

Now we can construct, inductively, pairwise disjoint embedded 2-dimensional open discs $E_i$ such that $\partial E_i = \alpha_i \cup \beta'_i$ for any $i$. Also, there are smooth unknotted paths $\gamma_i$ in $V - \bigcup_{i \geq 1} B_i - \bigcup_{i \geq 1} E_i$ joining $\alpha_i(0)$ and $\alpha_i(1)$, near $\alpha_i$ and such that each crossing of $\bigcup_{i \geq 1} \gamma_i \cup \bigcup_{i \geq 1} \beta'_i$ is an overcrossing. Thus, there are pairwise disjoint small neighbourhoods $U_i$ of $E_i$ in $V - \bigcup_{i \geq 1} B_i - \bigcup_{i \geq 1} \gamma_i$. Then, there is an isotopy $\theta: \bigcup_{i \geq 1} \alpha_i \times [0,1] \to \bigcup_{i \geq 1} U_i$ with $\theta_0$ equal to the identity and $\theta_1$ equal to $f_3^{-1} \circ f_2^{-1} \circ f_1^{-1} \circ f$.

Now, the proof follows as in Lemma 4.4 of [5].

**COROLLARY.** There is no normal subgroup between $\text{Diff}^\Omega_c(R^3)$ and $\text{Diff}^\Omega_f(R^3)$.

**REFERENCES**


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