

ON INNER PRODUCT SPACES OVER DEDEKIND DOMAINS OF CHARACTERISTIC TWO

RICHARD C. WAGNER

ABSTRACT. Suppose D is a Dedekind domain of characteristic 2 and (M, φ) is an inner product space, i.e. M is a finitely generated projective D module supplied with a nonsingular symmetric bilinear form φ . It is shown that (M, φ) is determined up to isometry by the extension of φ to $F \otimes_D M$, where F is the quotient field of D , and the value module $\mathcal{Q}(M)$ of all $\varphi(m, m)$ for m in M . In particular, a hyperbolic space $\mathbf{H}(M)$ is completely determined by the rank of the finitely generated projective module M . As consequences, genera coincide with isometry classes, and if N_1 and N_2 are isometric nonsingular submodules of (M, φ) such that $\mathcal{Q}(N_1^\perp) = \mathcal{Q}(N_2^\perp)$, then N_1^\perp and N_2^\perp are isometric. Also, given an F inner product space (V, φ) and a $D^{(2)}$ submodule P of D , a necessary and sufficient condition is given for the existence of a D inner product space (M, Ψ) such that $(FM, \Psi) \cong (V, \varphi)$ and $\mathcal{Q}(M) = P$.

1. Preliminaries. Unless otherwise stated, D is a Dedekind domain of characteristic 2 with quotient field F and all D modules will be left D modules. $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ are fractional ideals of D , and $\mathfrak{A}^{-1} = \{x \in F \mid x\mathfrak{A} \subseteq D\}$ is the inverse of the fractional ideal \mathfrak{A} .

If M is a finitely generated projective D module, an inner product on M is a nonsingular symmetric bilinear form φ on M . In this case (M, φ) will be called an inner product space. If FM denotes the F vector space $F \otimes_D M$, then φ has a unique extension to a nonsingular symmetric bilinear form on FM . We also use φ to denote this extension, so (FM, φ) is also an inner product space.

We use $D^{(2)}$ ($F^{(2)}$) to denote the subring of D (F) of all squares. If (M, φ) and (FM, φ) are inner product spaces, then $\mathcal{Q}(M) = \{\varphi(m, m) \mid m \in M\}$ and $\mathcal{Q}(FM) = \{\varphi(x, x) \mid x \in FM\}$ are subgroups of D and F , respectively. They become $D^{(2)}$ or $F^{(2)}$ submodules when we define $a^2 \cdot \varphi(m, m) = \varphi(am, am)$ for a in D or F , or they become D or F submodules when we define $a \circ \varphi(m, m) = \varphi(am, am)$. $\mathcal{Q}(M)$ and $\mathcal{Q}(FM)$ are called the value modules of the respective inner product spaces. We note that $q_\varphi: M \rightarrow \mathcal{Q}(M)$, defined by $q_\varphi(m) = \varphi(m, m)$, is a D module homomorphism when $\mathcal{Q}(M)$ is viewed as a D module.

If P is a projective module over an arbitrary commutative ring and k is an arbitrary symmetric bilinear form on P , we let $P^* = \text{Hom}_D(P, D)$ and define the inner product space $\mathbf{M}(P) = (P \oplus P^*, h_p)$ by $h_p(x + f, y + g) = k(x, y) + g(x) + f(y)$, $x, y \in P, f, g \in P^*$. Any inner product space isometric to a $\mathbf{M}(P)$ is called

Received by the editors March 23, 1982.

1980 *Mathematics Subject Classification*. Primary 10C02, 10C03, 15A63; Secondary 13B30, 13C10, 13F30.

Key words and phrases. Symmetric bilinear forms, value modules, hyperbolic and metabolic spaces, Witt cancellation, genera, Dedekind and discrete valuation rings.

©1985 American Mathematical Society
0002-9939/85 \$1.00 + \$.25 per page

a metabolic (or split) space. It is easy to see that an inner product space (M, φ) is metabolic if and only if there is a submodule N of M which is a direct summand of M and N is precisely equal to its orthogonal complement N^\perp . If the symmetric bilinear form k is identically zero, then we denote $\mathbf{M}(P)$ by $\mathbf{H}(P)$. Any inner product space isometric to a $\mathbf{H}(P)$ is said to be a hyperbolic space. We note that $(0) \oplus P^*$ is totally isotropic in both $\mathbf{M}(P)$ and $\mathbf{H}(P)$, while $P \oplus (0)$ is also totally isotropic in $\mathbf{H}(P)$. If P is a free D module on one generator then $\mathbf{H}(P)$ is called a hyperbolic plane, and if $x, y \in \mathbf{H}(P)$ are such that $h_p(x, x) = 0 = h_p(y, y)$ and $h_p(x, y) = 1$, then x, y is called a hyperbolic basis for the plane.

Proofs for the following over an arbitrary commutative ring can be found in [4].

1.1. *If P is a projective D module and $P = P_1 \oplus P_2$, then $\mathbf{M}(P) \cong \mathbf{M}(P_1) \perp \mathbf{M}(P_2)$. In particular, $\mathbf{H}(P) \cong \mathbf{H}(P_1) \perp \mathbf{H}(P_2)$.*

1.2. *If (P, φ) is an inner product space then $(P, \varphi) \perp (P, -\varphi) \cong \mathbf{M}(P)$.*

We now introduce a mapping similar to that of A. Roy [9]; also see [12]. Suppose (X, φ) is an inner product space where $X = \mathbf{M}(P) \perp Q$. If $\alpha: P \rightarrow Q$ is a D module homomorphism we define

$$E_\alpha: \mathbf{M}(P) \perp Q \rightarrow \mathbf{M}(P) \perp Q$$

by

$$\begin{aligned} E_\alpha(x) &= x + \alpha(x), & x \in P, \\ E_\alpha(f) &= f, & f \in P^*, \\ E_\alpha(y) &= y - \alpha^*(y), & y \in Q. \end{aligned}$$

Here $\alpha^*: Q \rightarrow P^*$ is defined by

$$\alpha^* = 'a \cdot d_\varphi,$$

where $d_\varphi: Q \rightarrow Q^*$, $'\alpha: Q^* \rightarrow P^*$ are defined by $d_\varphi(z) = \varphi(\ , z)$ and $'\alpha$ is the dual of α .

The following is easily established.

- 1.3(i) E_α restricted to Q is an isometry onto $E_\alpha(Q)$.
(ii) $\mathbf{M}(P) \perp Q = E_\alpha(\mathbf{M}(P)) \perp E_\alpha(Q)$.

1.4 THEOREM. *Given $\mathbf{M}(P)$ and (Q, φ) inner product spaces with $\mathcal{Q}(\mathbf{M}(P)) \subseteq \mathcal{Q}(Q)$, we have*

$$\mathbf{M}(P) \perp (Q, \varphi) \cong \mathbf{H}(P) \perp (Q, \varphi).$$

PROOF. We have a D module homomorphism $h: Q \rightarrow \mathcal{Q}(Q)$ defined by $h(x) = \varphi(x, x)$. Because $\mathcal{Q}(Q)$ is a torsion free, finitely generated D module, it is a projective D module. Consequently, $Q = Q_1 \oplus K$, where K is the kernel of h . Let $h^{-1}: \mathcal{Q}(Q) \rightarrow Q_1$ denote the inverse of h restricted to Q_1 . If k_p is the symmetric bilinear form on P in $\mathbf{M}(P)$ we let $k: P \rightarrow \mathcal{Q}(\mathbf{M}(P))$ be the D module homomorphism defined by $k(x) = k_p(x, x)$, and let $\alpha: P \rightarrow Q$ be defined by $\alpha = h^{-1}k$. We note that if $x \in P$ then $k(x) = h(\alpha(x))$. Therefore $E_\alpha(x) = x + \alpha(x)$ is isotropic and, consequently, $E_\alpha(P)$ is totally isotropic. By 1.3,

$$\mathbf{M}(P) \perp Q = E_\alpha(\mathbf{M}(P)) \perp E_\alpha(Q),$$

and $E_\alpha(\mathbf{M}(P))$ is isometric to $\mathbf{H}(P)$ and $E_\alpha(Q)$ is isometric to Q . \square

2. Cancellation and classification. The following argument, due to A. Frohlich [2], shows that Witt cancellation over Dedekind domains of characteristic zero is not always possible.

Let \mathfrak{A} and \mathfrak{B} be fractional ideals of the Dedekind domain D . Because

$$D \oplus \mathfrak{A} \cong \mathfrak{B}^{-1} \oplus \mathfrak{B}\mathfrak{A} \quad \text{and} \quad \mathfrak{B} \oplus \mathfrak{B}\mathfrak{A} \cong D \oplus \mathfrak{B}^2\mathfrak{A},$$

we have

$$\begin{aligned} \mathbf{H}(D) \perp \mathbf{H}(\mathfrak{A}) &\cong \mathbf{H}(D \oplus \mathfrak{A}) \cong \mathbf{H}(\mathfrak{B}^{-1} \oplus \mathfrak{B}\mathfrak{A}) \cong \mathbf{H}(\mathfrak{B}^{-1}) \perp \mathbf{H}(\mathfrak{B}\mathfrak{A}) \\ &\cong \mathbf{H}(\mathfrak{B}) \perp \mathbf{H}(\mathfrak{B}\mathfrak{A}) \cong \mathbf{H}(\mathfrak{B} \oplus \mathfrak{B}\mathfrak{A}) \cong \mathbf{H}(D \oplus \mathfrak{B}^2\mathfrak{A}) \\ &\cong \mathbf{H}(D) \perp \mathbf{H}(\mathfrak{B}^2\mathfrak{A}). \end{aligned}$$

If the characteristic of D is zero then Frohlich shows that $\mathbf{H}(\mathfrak{A})$ is not always isometric to $\mathbf{H}(\mathfrak{B}^2\mathfrak{A})$. Thus $\mathbf{H}(D)$ cannot be cancelled from both sides.

The situation for Dedekind domains of characteristic two is quite different, as the following illustrates.

2.1 THEOREM. *Let D be a Dedekind domain of characteristic two and M a finitely generated, projective D module. If n is the rank of M then $\mathbf{H}(M) \cong \mathbf{H}(D^n)$.*

PROOF. Because M can be expressed as a direct sum of fractional ideals of D , it suffices to show that $\mathbf{H}(\mathfrak{A}) \cong \mathbf{H}(D)$ when \mathfrak{A} is a fractional ideal of D . Because $\mathfrak{A}^{-1} = \text{Hom}_D(\mathfrak{A}, D)$, $\mathbf{H}(\mathfrak{A})$ has $\mathfrak{A} \oplus \mathfrak{A}^{-1}$ for its module and $h_{\mathfrak{A}}: \mathfrak{A} \times \mathfrak{A}^{-1} \rightarrow D$ for its symmetric bilinear form. By [1, Theorem 22.11], $\mathfrak{A} \oplus \mathfrak{A}^{-1}$ is isomorphic to a free D module of rank two, so there are x and y in $\mathfrak{A} \oplus \mathfrak{A}^{-1}$ so that $\mathfrak{A} \oplus \mathfrak{A}^{-1} = Dx \oplus Dy$. Because $\mathcal{L}(\mathbf{H}(\mathfrak{A})) = (0)$, x and y must both be isotropic, and because $\mathbf{H}(\mathfrak{A})$ is nonsingular, $h_{\mathfrak{A}}(x, y)$ must be a unit of D . Therefore $\mathbf{H}(\mathfrak{A}) \cong \mathbf{H}(D)$. \square

The preceding result raises the possibility that cancellation of hyperbolic direct summands is possible over Dedekind domains of characteristic two. In order to prove this, we need the following lemmas.

2.2 LEMMA. *Let (X, φ) be an inner product space over D and $X = H \perp M$, where H is a hyperbolic plane with hyperbolic basis x, y . If $z \in M$ and $H' = Dx \oplus D(y + z)$ then there is a submodule M' of X isometric to M such that $X = H' \perp M'$.*

2.3 LEMMA. *Let (X, φ) be an inner product space over D such that $H_1 \perp M = X = H_2 \perp N$, where H_1 and H_2 are hyperbolic planes. Suppose H_1 and H_2 have hyperbolic bases x, y and w, z , respectively. If $\varphi(x, w) = u$, an invertible element of D , then M and N are isometric.*

Proofs can be found in [11]. Also see [7, pp. 256–257 and 3, Theorem 40].

The following establishes cancellation of hyperbolic spaces.

2.4 THEOREM. *Suppose (X, φ) is an inner product space over D such that $H_1 \perp M = X = H_2 \perp N$, where H_1 and H_2 are isometric hyperbolic spaces. Then M and N are isometric.*

PROOF. Because of Theorem 2.1 it suffices to prove the theorem when H_1 and H_2 are hyperbolic planes. Let x, y and w, z be hyperbolic bases for H_1 and H_2 , respectively. If $\varphi(H_1, H_2) = (0)$ then $H_1 \subseteq N$ and there is a submodule N' of N such that $N = H_1 \perp N'$. Now

$$H_1 \perp M = X = H_1 \perp (H_2 \perp N'),$$

so by Lemma 2.3, M and $H_2 \perp N'$ are isometric. But $H_2 \perp N'$ and $H_1 \perp N'$ are isometric, so M and N are isometric.

Thus we can assume $\varphi(H_1, H_2) \neq (0)$ and, without loss of generality, $\varphi(w, H_1) \neq (0)$. We wish to reduce to the case in which $\varphi(w, H_1) = D$. Suppose $\varphi(w, H_1) = \mathfrak{A} \neq D$. We have $z = h + m$, where $h \in H_1$, $m \in M$ and m must be isotropic. But $\varphi(w, z) = 1$, so $\mathfrak{A} + \varphi(w, m)D = D$. Let $\varphi(w, x) = a$, $\varphi(w, y) = b$ and $\varphi(w, m) = c$. Then $Da + Db + Dc = D$. Because $\mathfrak{A} \neq (0)$, either $a \neq 0$ or $b \neq 0$; say $a \neq 0$. Let $\bar{D} = D/Da$ and let \bar{x} represent the image of $x \in D$ in \bar{D} . But \bar{D} is a finite direct sum of fields and local principal ideal rings [14, p. 278] and, consequently, has stable range one, i.e. because $\bar{D}b + \bar{D}c = \bar{D}$ there is $\bar{r} \in \bar{D}$ such that $\bar{D}(\bar{b} + \bar{r}c) = \bar{D}$. Therefore $Da + D(b + rc) = D$. Now $H_3 = Dx \oplus D(y + rm)$ is a hyperbolic plane and by Lemma 2.2 there is a submodule M' of X , isometric to M , such that $X = H_3 \perp M'$ and $\varphi(w, H_3) = D$. Thus we can assume from the beginning that $\varphi(w, H_1) = D$. Now define a D module homomorphism $f: Dx \oplus Dy \rightarrow D$ by $f(u) = \varphi(w, u)$, so f is onto D . Suppose $x' \in Dx \oplus Dy$ with $f(x') = 1$. Then x' is isotropic and $Dx \oplus Dy = Dx' \oplus K$, where K is the kernel of f . Now K must be free of rank 1, say $K = Dy'$. But y' is isotropic, so by multiplying y' by an invertible element of D , if necessary, x', y' is a hyperbolic basis for $H_1 = Dx \oplus Dy$. Because $\varphi(w, x') = 1$, we have M and N isometric by Lemma 2.3. \square

We are now in a position to characterize inner product spaces by their extensions to the quotient field F of D and their value modules. The analogous result for local rings of characteristic two has been established by C. H. Sah [10, p. 268]. Also see [7, p. 259].

2.5 THEOREM. *Let (M, φ) and (N, Ψ) be inner product spaces such that (FM, φ) and (FN, Ψ) are isometric. Then (M, φ) and (N, Ψ) are isometric if and only if $\mathcal{Q}(M) = \mathcal{Q}(N)$.*

PROOF. If (M, φ) and (N, Ψ) are isometric it is clear that $\mathcal{Q}(M) = \mathcal{Q}(N)$. For the converse, we assume that (FM, φ) and (FN, Ψ) are isometric and $\mathcal{Q}(M) = \mathcal{Q}(N)$.

Let $X = M \perp N \perp N$ be provided with the direct sum of the corresponding inner products. By Theorems 1.4 and 2.1,

$$X \cong M \perp \mathbf{M}(N) \cong M \perp \mathbf{H}(N) \cong M \perp \mathbf{H}(D^n),$$

where n is the rank of N .

On the other hand, $FM \perp FN$ has a totally isotropic subspace V of dimension n such that $V = V^\perp$. Let $M_0 = V \cap (M \perp N)$, so M_0 is a pure submodule of $M \perp N$ and, consequently, a totally isotropic direct summand of $M \perp N$ of rank n . Suppose $M \perp N = M_0 \oplus M_1$, so M_1 has rank n . Then M_1 must be anisotropic, for otherwise $FM \perp FN$ would have a totally isotropic subspace of dimension greater than n .

Because $V = V^\perp$, we have $M_0 = M_0^\perp$ and, consequently, $M \perp N \cong \mathbf{M}(M_1)$. If $M_1 \cong D^{n-1} \oplus \mathfrak{A}$, where \mathfrak{A} is an integral ideal, then

$$\begin{aligned} \mathbf{H}(\mathfrak{A}^{-1}) \perp X &= \mathbf{H}(\mathfrak{A}^{-1}) \perp M \perp N \perp N \\ &\cong \mathbf{H}(\mathfrak{A}^{-1}) \perp \mathbf{M}(D^{n-1}) \perp \mathbf{M}(\mathfrak{A}) \perp N \cong \mathbf{M}(D^{n+1}) \perp N, \end{aligned}$$

because $\mathbf{H}(\mathfrak{A}^{-1}) \perp \mathbf{M}(\mathfrak{A}) \cong \mathbf{M}(D^2)$. But $\mathcal{Q}(\mathbf{M}(D^{n+1})) = \mathcal{Q}(M) = \mathcal{Q}(N)$, so by Theorem 1.4,

$$\mathbf{M}(D^{n+1}) \perp N \cong \mathbf{H}(D^{n+1}) \perp N.$$

Thus

$$\mathbf{H}(D^{n+1}) \perp N \cong \mathbf{H}(\mathfrak{A}^{-1}) \perp X \cong \mathbf{H}(D^{n+1}) \perp M.$$

By Theorem 2.4, M and N are isometric. \square

In particular, a D inner product space (M, φ) is hyperbolic if and only if $\mathcal{Q}(M) = (0)$.

If \mathfrak{P} is a maximal ideal of D then the completion of F with respect to the \mathfrak{P} -adic valuation will be denoted by $\hat{F}_{\mathfrak{P}}$ and the valuation ring by $\hat{D}_{\mathfrak{P}}$. If (M, φ) is a D inner product space, we let $\hat{M}_{\mathfrak{P}} = \hat{D}_{\mathfrak{P}} \otimes_D M$ and let $\varphi_{\mathfrak{P}}$ be the extension of φ to $\hat{M}_{\mathfrak{P}}$.

2.6 COROLLARY. *If (M, φ) and (N, Ψ) are two inner product spaces over D , (FM, φ) is isometric to (FN, Ψ) and $(\hat{M}_{\mathfrak{P}}, \varphi_{\mathfrak{P}})$ is isometric to $(\hat{N}_{\mathfrak{P}}, \Psi_{\mathfrak{P}})$ for all maximal ideals \mathfrak{P} of D , then (M, φ) and (N, Ψ) are isometric.*

PROOF. If \mathfrak{P} is a maximal ideal of the Dedekind domain D then $D^{(2)}$ is also a Dedekind domain with quotient field $F^{(2)}$ and maximal ideal $\mathfrak{P}^{(2)}$. We denote the completion of $F^{(2)}$ with respect to the $\mathfrak{P}^{(2)}$ -adic valuation by $\hat{F}_{\mathfrak{P}^{(2)}}$ and the valuation ring by $\hat{D}_{\mathfrak{P}^{(2)}}$. It is easy to see that $\hat{F}_{\mathfrak{P}^{(2)}}$ is the subfield of $\hat{F}_{\mathfrak{P}}$ of squares and $\hat{D}_{\mathfrak{P}^{(2)}}$ is the subring of $\hat{D}_{\mathfrak{P}}$ of squares.

If $\mathcal{Q}(M)$ is viewed as a $D^{(2)}$ module, by [8, Theorem 5.3] we have

$$\mathcal{Q}(M) = \cap [\mathcal{Q}(M) \otimes F^{(2)} \cap \mathcal{Q}(M) \otimes \hat{D}_{\mathfrak{P}^{(2)}}],$$

where the tensor products are over $D^{(2)}$ and the intersection ranges over all maximal ideals \mathfrak{P} of D . The corollary now follows from the theorem and the observation that

$$\begin{aligned} \mathcal{Q}(M) \otimes_{D^{(2)}} F^{(2)} &= \mathcal{Q}(M \otimes_D F) = \mathcal{Q}(FM), \\ \mathcal{Q}(M) \otimes_{D^{(2)}} \hat{D}_{\mathfrak{P}^{(2)}} &= \mathcal{Q}(M \otimes_D \hat{D}_{\mathfrak{P}}) = \mathcal{Q}(\hat{M}_{\mathfrak{P}}). \quad \square \end{aligned}$$

2.7 COROLLARY. *Let (X, φ) be a D inner product space and M and N nonsingular isometric submodules so that $M \perp M^\perp = X = N \perp N^\perp$. If $\mathcal{Q}(M^\perp) = \mathcal{Q}(N^\perp)$ then M^\perp and N^\perp are isometric.*

PROOF. We have

$$FM \perp FM^\perp = FX = FN \perp FN^\perp.$$

Because $\mathcal{Q}(M^\perp) = \mathcal{Q}(N^\perp)$, we also have $\mathcal{Q}(FM^\perp) = \mathcal{Q}(FN^\perp)$. By [12, Theorem 2.3], FM^\perp and FN^\perp are isometric and, consequently, M^\perp and N^\perp are isometric. \square

In [12, Theorem 2.3] it was shown that when D is a field of characteristic 2, M^\perp being anisotropic is sufficient to guarantee cancellation. We give an example to show that this is not always the case for discrete valuation rings of characteristic 2.

Let $R = K[[\pi]]$ be the ring of formal power series in the indeterminate π over the field $K = Z_2(t)$ of rational functions in the indeterminate t over the field Z_2 of two elements. We claim that

$$\langle 1 \rangle \perp \begin{bmatrix} \pi^2, 1 \\ 1, t \end{bmatrix} \cong \langle 1 \rangle \perp \begin{bmatrix} \pi^2, u \\ u, t+1 \end{bmatrix},$$

where $u = 1 + \pi$.

If, on the left-hand side, x is a basis for the one-dimensional direct summand and y, z is a basis for the two-dimensional direct summand, then on the right-hand side, $x + w$ is a basis for the one-dimensional summand and $y + \pi w, z + x + w$, where $w = y + \pi x$, is a basis for the two-dimensional summand. It is easy to see that the two-dimensional summands are anisotropic and they cannot be isometric because they have different value models. Thus the one-dimensional summand $\langle 1 \rangle$ cannot be cancelled.

The existence question remains; namely, given an F inner product space (V, φ) and a $D^{(2)}$ submodule P of D , is there a D inner product space (M, Ψ) such that $(FM, \Psi) \cong (V, \varphi)$ and $\mathcal{Q}(M) = P$?

To answer this question, we introduce some terminology. If (V, φ) is an F inner product space, we say that a D lattice M in V is integral if φ restricted to M takes values only in D , and we say that it is nonsingular if, in addition, φ restricted to M is nonsingular.

Given (M, φ) , an anisotropic D inner product space with $Q = \mathcal{Q}(M)$, a $D^{(2)}$ submodule of D , we define

$$B_\varphi: Q \times Q \rightarrow D^{(2)} \quad \text{by} \quad B_\varphi(a, b) = \varphi(x, y)^2,$$

where $a = \varphi(x, x)$, $b = \varphi(y, y)$ for some $x, y \in M$. It is easy to see that B_φ is a well-defined symmetric bilinear form with $B_\varphi(a, a) = a^2$ when $a \in \mathcal{Q}(M)$. Furthermore:

- 2.8 LEMMA. (i) $(\mathcal{Q}(M), B_\varphi)$ is a $D^{(2)}$ inner product space.
(ii) Two anisotropic D inner product spaces (M, φ) and (N, Ψ) are isometric if and only if $\mathcal{Q}(M) = \mathcal{Q}(N)$ and $B_\varphi = B_\Psi$.
(iii) If L is a D lattice in (FM, φ) , we associate to it the $D^{(2)}$ lattice $\mathcal{Q}(L)$ of $(\mathcal{Q}(FM), B_\varphi)$. This establishes a one-to-one correspondence between integral D lattices in (FM, φ) and integral $D^{(2)}$ lattices in $(\mathcal{Q}(FM), B_\varphi)$ and between corresponding nonsingular lattices.

PROOF. If $a \in D^{(2)}$, we use \sqrt{a} to denote the unique element b of D such that $b^2 = a$. If $q_\varphi: M \rightarrow \mathcal{Q}(M)$ is the D module homomorphism defined by $q_\varphi(m) = \varphi(m, m)$, then $B_\varphi(q_\varphi(m), q_\varphi(n)) = \varphi(m, n)^2$ for $m, n \in M$.

To prove (i) assume $f \in \text{Hom}_{D^{(2)}}(\mathcal{Q}(M), D^{(2)})$. Then $f^*(m) = \sqrt{f(\varphi(m, m))}$ is in $\text{Hom}_D(M, D)$ so there is $x \in M$ such that $f^* = \varphi(x, \)$ and, consequently, $f = B_\varphi(q_\varphi(x), \)$.

In order to prove (ii) assume first that (M, φ) and (N, Ψ) are isometric. Then we clearly have $\mathcal{Q}(M) = \mathcal{Q}(N)$ and, if $f: M \rightarrow N$ is an isometry, $\varphi(m, n) = \Psi(f(m), f(n))$ for $m, n \in M$. Consequently,

$$B_\varphi(q_\varphi(m), q_\varphi(n)) = B_\Psi(q_\Psi(f(m)), q_\Psi(f(n))),$$

so $B_\varphi = B_\Psi$. Conversely, assume $\mathcal{Q}(M) = \mathcal{Q}(N)$ and $B_\varphi = B_\Psi$. Because (N, Ψ) is anisotropic, q_Ψ is invertible and $q_\Psi^{-1}q_\varphi$ is easily seen to be the required isometry.

It is easy to verify the first two assertions of (iii). To prove the third, assume B_φ is nonsingular on $\mathcal{Q}(M)$ and we are given $f \in \text{Hom}_{D^{(2)}}(L, D)$. Then $f^* = (fq_\varphi^{-1})^2 \in \text{Hom}_{D^{(2)}}(\mathcal{Q}(L), D^{(2)})$ and there is $a \in \mathcal{Q}(L)$ such that $f^* = B_\varphi(a, \cdot)$. But then $f = \varphi(q_\varphi^{-1}(a, \cdot))$ and, consequently, φ is nonsingular on L . Conversely, if we assume φ is nonsingular on L and we are given $f^* \in \text{Hom}_{D^{(2)}}(\mathcal{Q}(L), D^{(2)})$, then $f = \sqrt{f^*q_\varphi} \in \text{Hom}_D(L, D)$ and there is $x \in L$ such that $f = \varphi(x, \cdot)$. Then $f^* = B_\varphi(q_\varphi(x), \cdot)$ so B_φ is nonsingular on $\mathcal{Q}(L)$. \square

Given an anisotropic D inner product space (M, φ) , we call $(\mathcal{Q}(M), B_\varphi)$ the Wall invariant of (M, φ) . See [13, Lemma 3.4.2]. Also, recall from [12, p. 375] that $V = H \perp V_1 \perp V_1 \perp V_0$ is said to be a Witt decomposition of the F inner product space (V, φ) if H is a hyperbolic space, V_1 and V_0 are anisotropic and $\mathcal{Q}(V_1) \cap \mathcal{Q}(V_0) = (0)$. Also see [4, p. 33 and 5, p. 66].

We are now in a position to answer the existence question.

2.9 THEOREM. *Given an F inner product space (V, φ) with Witt decomposition $V = H \perp V_1 \perp V_1 \perp V_0$ and a $D^{(2)}$ submodule P of D , there is a D inner product space (M, Ψ) such that $\mathcal{Q}(M) = P$ and $(FM, \Psi) \cong (V, \varphi)$ if and only if*

- (i) $F^{(2)}P = \mathcal{Q}(V)$,
- (ii) *there is a maximal nonsingular $D^{(2)}$ lattice L in the Wall invariant $(\mathcal{Q}(V_0), B_\varphi)$ of (V_0, φ) which is contained in P such that $(P \cap \mathcal{Q}(V_0))/L$ is a torsion module and is a direct sum of no more than $\frac{1}{2} \dim H$ cyclic $D^{(2)}$ modules.*

Moreover, the ideal class of M is equal to the ideal class of L when it is viewed as a D module.

PROOF. We first assume that there is a D inner product space (M, Ψ) with $(FM, \Psi) \cong (V, \varphi)$ and $\mathcal{Q}(M) = P$. Then we clearly have $F^{(2)}P = \mathcal{Q}(V)$. Because D is a Dedekind domain, $M = \mathbf{M}(M_1) \perp M_0$, where M_0 is anisotropic. Then

$$FM(M_1) \perp FM_0 \cong V = H \perp V_1 \perp V_1 \perp V_0,$$

so $FM_0 \cong V_0$ by [12, Theorem 2.5]. Consequently, $\mathcal{Q}(FM_0) = \mathcal{Q}(V_0)$ and $B_\Psi = B_\varphi$ by Lemma 2.8. Therefore $(\mathcal{Q}(M_0), B_\Psi)$ is a nonsingular $D^{(2)}$ lattice on $(\mathcal{Q}(V_0), B_\varphi)$ which is contained in P . Suppose L is a nonsingular lattice in $(\mathcal{Q}(V_0), B_\varphi)$ containing $\mathcal{Q}(M_0)$ and contained in P . Then (L, B_φ) is a $D^{(2)}$ inner product space so there is a submodule L_0 of L such that

$$L = L_0 \perp \mathcal{Q}(M_0), \quad \text{so} \quad F^{(2)}L = F^{(2)}L_0 \perp F^{(2)}\mathcal{Q}(M_0).$$

But $F^{(2)}\mathcal{Q}(M_0) = \mathcal{Q}(FM_0) = \mathcal{Q}(V_0)$ so $F^{(2)}L_0 = (0)$ and $L = \mathcal{Q}(M_0)$. Therefore $\mathcal{Q}(M_0)$ is a maximal nonsingular lattice in the Wall invariant $(\mathcal{Q}(V_0), B_\varphi)$ of (V_0, φ) .

By Theorem 1.4, we can assume that M_1 has no direct summand N such that $\mathcal{Q}(N) \neq (0)$ and $\mathcal{Q}(N) \subseteq \mathcal{Q}(M_0)$. Let

$$N_2 = \{m \in M_1 \mid \Psi(m, m) \in D \cap \mathcal{Q}(V_0)\}.$$

Then N_2 is a pure submodule of M_1 so there is a submodule N_1 of M_1 so that $M_1 = N_1 \oplus N_2$. By Theorem 1.1, $\mathbf{M}(M_1) \cong \mathbf{M}(N_1) \perp \mathbf{M}(N_2)$. Also, $P \cap \mathcal{Q}(V_0) = \mathcal{Q}(N_2) + \mathcal{Q}(M_0)$ and $\mathcal{Q}(N_2) \subseteq F^{(2)}\mathcal{Q}(M_0)$. By the Invariant Factor Theorem [1, p. 150], there are elements m_1, \dots, m_n in $\mathcal{Q}(M_0)$ such that

$$\mathcal{Q}(M_0) = \mathfrak{S}_1 m_1 \oplus \cdots \oplus \mathfrak{S}_n m_n \quad \text{and} \quad \mathcal{Q}(N_2) = \mathfrak{E}_1 \mathfrak{S}_1 m_1 \oplus \cdots \oplus \mathfrak{E}_n \mathfrak{S}_n m_n,$$

where the \mathfrak{S}_i and \mathfrak{E}_i are fractional ideals of $D^{(2)}$ and $\mathfrak{E}_1 \supseteq \mathfrak{E}_2 \supseteq \cdots \supseteq \mathfrak{E}_n$ are called the invariant factors. If k is the rank of $\mathcal{Q}(N_2)$ as a D module, $\mathfrak{E}_i = (0)$ for $i > k$. Because of our observation regarding direct summands of M_1 , \mathfrak{E}_i is not an integral ideal for $i \leq k$. Therefore $\mathfrak{S}_i m_i \subseteq \mathfrak{E}_i \mathfrak{S}_i m_i$ for $i \leq k$ and

$$(P \cap \mathcal{Q}(V_0))/\mathcal{Q}(M_0) \cong \mathfrak{E}_1 \mathfrak{S}_1 / \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{E}_k \mathfrak{S}_k / \mathfrak{S}_k.$$

By the Invariant Factor Theorem and Theorem 4.12 of [8, p. 48], each $\mathfrak{E}_i \mathfrak{S}_i / \mathfrak{S}_i$ is cyclic. Also, $F\mathbf{M}(N_2) \perp FM_0$ is isometric to a subspace of $H \perp V_0$ and $\dim FM_0 = \dim V_0$ so $2k \leq \text{rank } \mathbf{M}(N_2) \leq \dim H$.

To prove the converse, we assume (i) and (ii) are satisfied. We first consider the case in which $H = (0)$ and $V_0 = (0)$, so $V = V_1 \perp V_1$. We proceed by induction on the rank of P as a $D^{(2)}$ module. If the rank of P is one, we can identify P with a fractional ideal of $D^{(2)}$ which we will view as a D module using $r \circ a = r^2 a$ for $r \in D$ and $a \in P$. Define $h: P \times P \rightarrow D$ by $h(a, b) = \sqrt{ab}$. Then h is a symmetric bilinear form and $\mathcal{Q}(\mathbf{M}(P)) = P$. Because $\mathcal{Q}(F^{(2)}\mathbf{M}(P)) = F^{(2)}P = \mathcal{Q}(V_1)$, $F^{(2)}\mathbf{M}(P) \cong V_1 \perp V_1$ by Theorem 2.5(iii) [12, p. 382]. In the case that P has rank greater than one, we can represent P as a direct sum of two nonzero submodules, use the induction hypothesis on each, and piece together the results to prove the assertion for P .

We now consider the general case. By Lemma 2.8(iii) there is a nonsingular D lattice M_0 in V_0 such that $\mathcal{Q}(M_0) = L$, so $(FM_0, \varphi) = (V_0, \varphi)$. Now $P \cap \mathcal{Q}(V_0)$ is a pure submodule of P so $P = (P \cap \mathcal{Q}(V_0)) \oplus Q$ for some $D^{(2)}$ submodule Q of P . Let $(\mathbf{M}(U_1), \Psi_1)$ be an F inner product space with $\mathcal{Q}(U_1) = F^{(2)}Q$ and having no hyperbolic part in its Witt decomposition. By the initial argument there is a D inner product space $(\mathbf{M}(M_1), \Psi_2)$ such that $(F\mathbf{M}(M_1), \Psi_2) \cong (\mathbf{M}(U_1), \Psi_1)$ and $\mathcal{Q}(\mathbf{M}(M_1)) = Q$. Because $(P \cap \mathcal{Q}(V_0))/L$ is a torsion module, $F^{(2)}L = F^{(2)}(P \cap \mathcal{Q}(V_0))$ and, consequently, $F^{(2)}L \oplus F^{(2)}Q = \mathcal{Q}(V)$. By Theorem 2.5 of [12, p. 382] we have

$$(2.10) \quad F\mathbf{M}(M_1) \perp FM_0 \cong V_1 \perp V_1 \perp V_0.$$

Now let S be a maximal submodule of L which is a direct summand of $P \cap \mathcal{Q}(V_0)$, so $P \cap \mathcal{Q}(V_0) = T \oplus S$ for some submodule T of $P \cap \mathcal{Q}(V_0)$. Because $S \subseteq L \subseteq P \cap \mathcal{Q}(V_0)$ we also have $L = (L \cap T) \oplus S$ and $(P \cap \mathcal{Q}(V_0))/L \cong T/(L \cap T)$. If T and $L \cap T$ had a common direct summand it would be a direct summand of $P \cap \mathcal{Q}(V_0)$, which would be contained in L , contradicting the maximality of S . By the Invariant Factor Theorem and (ii), $\text{rank}(L \cap T) = \text{rank } T \leq \frac{1}{2} \dim H$. Now let $(\mathbf{M}(U_2), \Psi_3)$ be an F inner product space which has no hyperbolic part in its Witt decomposition and $\mathcal{Q}(U_2) = F^{(2)}T$. By the initial argument there is a D inner product space $(\mathbf{M}(M_2), \Psi_4)$ such that $(F\mathbf{M}(M_2), \Psi_4) \cong (\mathbf{M}(U_2), \Psi_3)$ and $\mathcal{Q}(\mathbf{M}(M_2)) = T$. Because $F^{(2)}T \subseteq \mathcal{Q}(V_0)$,

$$F\mathbf{M}(M_2) \perp FM_0 \cong FH(M_2) \perp FM_0$$

by Theorem 1.4. Now let $n = \frac{1}{2} \dim H$ -rank T ,

$$(2.11) \quad M = \mathbf{H}(D^n) \perp \mathbf{M}(M_2) \perp \mathbf{M}(M_1) \perp M_0,$$

and let Ψ be the direct sum of the corresponding inner products. By (2.10) and (2.11), $(FM, \Psi) = (V, \varphi)$. Also by (2.11), $\mathcal{Q}(M)$ contains $L = \mathcal{Q}(M_0)$, $Q = \mathcal{Q}(\mathbf{M}(M_1))$ and $T = \mathcal{Q}(\mathbf{M}(M_2))$, so $\mathcal{Q}(M) = P$.

From (2.11) the ideal class of M is equal to that of M_0 . But $\mathcal{Q}(M_0) = L$ so the ideal class of M is equal to that of L when L is viewed as a D module. \square

I wish to thank the referee for suggestions.

REFERENCES

1. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience, New York, 1962.
2. A. Frohlich, *On the K -theory of unimodular forms over rings of algebraic integers*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 401–423.
3. I. Kaplansky, *Linear algebra and geometry*, Allyn and Bacon, Boston, Mass., 1969.
4. M. Knebusch, *Grothendieck und Witttringe von nichtausgearteten symmetrischen Bilinearformen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. 3 (1969/1970), 93–157.
5. J. Milnor, *Symmetric inner products in characteristic 2*, Prospects in Mathematics, Ann. of Math. Stud., No. 70, Princeton Univ. Press, Princeton, N.J., 1971, pp. 59–75.
6. J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Springer-Verlag, Berlin and New York, 1973.
7. O. T. O'Meara, *Introduction to quadratic forms*, Springer-Verlag, Berlin and New York, 1963.
8. I. Reiner, *Maximal orders*, Academic Press, London, 1975.
9. A. Roy, *Cancellation of quadratic forms over commutative rings*, J. Algebra **10** (1968), 286–298.
10. C. H. Sah, *A note on Hermitian forms over fields of characteristic 2*, Amer. J. Math. **86** (1964), 262–270.
11. R. C. Wagner, *Some Witt cancellation theorems*, Amer. J. Math. **94** (1972), 206–220.
12. _____, *On odd Hermitian forms*, J. Algebra **58** (1979), 370–398.
13. G. E. Wall, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*, J. Austral. Math. Soc. **3** (1963), 1–62.
14. O. Zariski and P. Samuel, *Commutative algebra*, Van Nostrand, Princeton, N.J., 1958.

DEPARTMENT OF MATHEMATICS, FAIRLEIGH DICKINSON UNIVERSITY, MADISON, NEW JERSEY 07940.