SYMBOLIC POWERS OF PRIME IDEALS
AND THEIR TOPOLOGY

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Abstract. For a prime ideal P of a commutative Noetherian ring R a necessary and sufficient condition is given to determine when the P-adic topology is equivalent, resp. linearly equivalent, to the P-symbolic topology. The last means that the symbolic Rees ring is a finitely generated module over the ordinary Rees ring of P. Then it is considered when the integral closure of all the powers of P are primary.

1. Introduction and main results. Let R denote a commutative Noetherian ring. For a prime ideal P of R we define $P^{(n)} = P^n R_P \cap R$, the n-th symbolic power of P. Note that $P^{(n)}$ is equal to the uniquely determined P-primary component of $P^n$. In [3, §7], R. Hartshorne writes: “A general question, whose solution is quite complicated, is to determine when the P-adic topology is equivalent to the P-symbolic topology.” Here the P-symbolic topology denotes the topology defined by the symbolic powers $P^{(n)}$, $n \geq 1$, of P. In the following we shall give a complete solution to this problem. For an arbitrary ideal I of R note that the sets $\text{Ass } R/I^n$ stabilize for large n (see M. Brodmann [1]). So we denote this stabilized set by $A^*(I)$.

Theorem 1. For a prime ideal P of a commutative Noetherian ring R the following conditions are equivalent:

(i) The P-adic topology is equivalent to the P-symbolic topology.
(ii) height($PR_Q + p/p$) < dim $R_Q/p$ for all $Q \in A^*(P) \setminus \{P\}$ and $p \in \text{Ass } R_Q$.

Here $R_Q$ denotes the $QR_Q$-adic completion of $R_Q$. A particular case of Theorem 1 has been shown by R. Hartshorne [3, Proposition 7.1]. Let P be a prime ideal satisfying condition (i) of the previous theorem. Then, for every integer $n \geq 1$ there is an integer $m \geq n$ such that $P^{(m)} \subseteq P^n$. Now the question is whether there exists an integer k such that $P^{(n+k)} \subseteq P^n$ for all $n \geq 1$. In this case we say that the P-adic topology is linearly equivalent to the P-symbolic topology. It is equivalent to the fact that the “symbolic” Rees ring $\bigoplus_{n \geq 0} P^{(n)}$ is a finitely generated graded module over $\bigoplus_{n \geq 0} P^n$, the ordinary Rees ring of P. For our next result let us recall the notion of analytic spread $a(I)$ of an ideal I of a local Noetherian ring $(R, M)$, i.e., $a(I) = \dim R(I) \otimes_R R/M$, where $R(I) = \bigoplus_{n \geq 0} I^n$ denotes the Rees ring of R with respect to I. Note that height $I \leq a(I) \leq \dim R$.
Theorem 2. For a prime ideal $P$ of a commutative Noetherian ring $R$ the following conditions are equivalent:

(i) The $P$-adic topology is linearly equivalent to the $P$-symbolic topology.

(ii) $a(PR_Q + p/p) < \dim R_Q/p$ for all $Q \in A^*(P) \setminus \{P\}$ and $p \in \text{Ass } R_Q$.

If one of these conditions is satisfied it follows:

(iii) $A^*_n(P) = \{P\}$.

Moreover, if $R$ is locally unmixed all of the above conditions are equivalent.

Here let $I_a$ denote the integral closure of an ideal $I$ of $R$, i.e., the set of elements $x \in R$ satisfying an equation $x^n + c_1x^{n-1} + \cdots + c_m = 0$, $c_i \in I^i$, $i = 1,\ldots,m$.

Then $\text{Ass } R/(I^n)_a$ stabilizes for large $n$, see L. J. Ratliff, Jr. [8]. This stabilized set is denoted by $A^*_n(I)$. A local ring $(R, M)$ is called unmixed if $\dim \hat{R}/p = \dim R$ for all $p \in \text{Ass } \hat{R}$, where $\hat{R}$ denotes the $M$-adic completion. $R$ is called locally unmixed if $R_Q$ is unmixed for all prime ideals $Q$ of $R$.

Corollary 1. Let $P$ denote a prime ideal of a locally unmixed Noetherian ring $R$. Then the following conditions are equivalent:

(i) There is an integer $k$ such that $P^n \subseteq P^k$ for all $n \geq 1$,

(ii) $a(PR_Q) < \dim R_Q$ for all $Q \in A^*(P) \setminus \{P\}$, and

(iii) $\text{Ass } P/(P^k)_a = \{P\}$ for all $n \geq 1$.

The question whether $\text{Ass } R/(P^n)_a = \{P\}$ for all $n \geq 1$ is a variation of the problem when does $\text{Ass } R/P^n = \{P\}$ for all $n \geq 1$, which has received some previous attention. It is a well-known fact that $\text{gr}_R(P)$ is a domain if and only if $\text{gr}_R((PR_P)_a)$ is a domain and $\text{Ass } P/P^k = \{P\}$ for all $n \geq 1$. Here $\text{gr}_R(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ denotes the form ring of $R$ with respect to an ideal $I$ of $R$.

Theorem 3. Let $P$ denote a prime ideal of a commutative Noetherian ring $R$. Then $\text{gr}_R(P)_a$ is a domain if and only if $\text{gr}_R((PR_P)_a)$ is a domain and $A^*_n(P) = \{P\}$.

If $R$ is a commutative Noetherian ring, by $R_{\text{red}}$ we denote the reduced ring of $R$, i.e., $R$ modulo its nilradical. Hence, $A^*_n(P) = \{P\}$ is, in a certain sense, a transitivity condition. Note that particular cases of Theorem 3 were obtained by C. Huneke [4]. Another result related to the question when does $A^*_n(P) = \{P\}$ has been shown by S. McAdam in [7].

2. Proof of Theorem 1. Before we shall give the proof, let us fix notation. For two ideals $I, J$ in a Noetherian ring $R$ we define $I: J = \bigcup_{k \geq 1} I: J^k$. Of course $I: \langle J \rangle = I: J^k$ for sufficiently large $k$.

(i) $\Rightarrow$ (ii): Assume the contrary. Then there are a prime ideal $Q \in A^*(P) \setminus \{P\}$ and a prime ideal $p \in \text{Ass } R_Q$ such that $\dim R_Q/(RQ + p) = 0$. By changing the notation we may assume $(R, M)$ a local ring and $\dim \hat{R}/(PR + p) = 0$ for some $p \in \text{Ass } \hat{R}$. Then $0 \neq 0: \langle p \rangle$, because $p \in \text{Ass } \hat{R}$. On the other hand

$$\text{gr}_R^k(p) = \text{gr}_R^k(p) = \text{gr}_R^k(p) = \text{gr}_R^k(p) \subseteq \text{gr}_R^k(p)$$

because $(p + \hat{R})^{2k} \subseteq p^{2k} + p^{2k} \subseteq (p + \hat{R})^{2k}$ for some integer $k \geq 1$ and because $p + \hat{R}$ is $M$-primary. Therefore it follows that $0 \neq 0: \langle p \rangle \subseteq \bigcap_{n \geq 1} p^n\hat{R}$, which is a contradiction to (i).
(ii) ⇒ (i): Set $E_{mn} = P^m + P^n/P^n$ for $m, n \geq 1$. Because $\text{Ass } E_{mn} \subseteq A^*(P) \setminus \{P\}$ it is enough to show the statement in any $R_Q, Q \in A^*(P) \setminus \{P\}$. That is, we may assume $R$ a local ring with $M$ its maximal ideal. Now we make an induction on $\dim R/P$. First assume $\dim R/P = 1$. For a fixed integer $n$, $E_{mn}$ becomes a decreasing sequence of modules of finite length. Hence, it becomes stationary. Set $E_n = P^{m(n)} + P^n$, where $m(n)$ is an integer such that $E_{mn} = E_{m(n),n}$ for all $m \geq m(n)$. It follows that $E_n = E_{n+1} + P^n$ for all $n$. Now assume $E_k \neq P^k$ for some $k$. So we may choose elements $x_n \in E_n \setminus P^n$ for all $n \geq k$ such that $x_{n+1} \equiv x_n \mod P^n$. Then $\{x_n\}$ defines a nonzero element of $\varprojlim_n E_n/P^n$. Therefore it follows that

$$0 \neq \bigcap_{n \geq 0} E_n = \varprojlim_n E_n/P^n \subseteq \varprojlim_n \hat{R}/P^n\hat{R} \cong \hat{R},$$

because $E_n/P^n = E_n/P^n \otimes_R \hat{R}$ is an $R$-module of finite length. Next we choose an element $x \in \bigcap_{n \geq 0} E_n$ such that $0:x = p$ for some associated prime ideal $p$ of $\hat{R}$. By the Artin-Rees Lemma there exists an integer $r \geq k$ such that $P^r\hat{R} \cap x\hat{R} \subseteq xP^r\hat{R}$. Since $x \in E_r$ and $E_r/P^r$ has finite length, $M^r x \subseteq P^r\hat{R} \cap x\hat{R} \subseteq xP^r\hat{R}$ for some $s \geq 1$. Then $M^s + p \subseteq P\hat{R} + p$, which contradicts (ii). Second suppose $\dim R/P > 1$. As above we consider $E_{mn}$. As the localizations of these modules in any prime ideal $Q \in A^*(P) \setminus \{P\}$ with $Q \neq M$ tend to zero by induction, they have finite length for $m$ large. Hence, $\text{Ass } E_{mn} \subseteq V(M)$ for $m$ large. Now the proof follows as in the case $\dim R/P = 1$. Hence we omit it. \( \square \)

**Remark.** It follows from the proof of Theorem 1 that we may replace (ii) in Theorem 1 by the following condition:

(ii)' $\text{height}(P\hat{R}_Q + p/p) < \dim R_Q/p$ for all $Q \in V(P) \setminus \{P\}$ and $p \in \text{Ass } R_Q$.

Here $V(P)$ denotes the set of prime ideals containing $P$.

3. Proof of Theorem 2. To this end we have to recall some notation. For an arbitrary ideal $I$ of $R$ we denote by $R(I) = \bigoplus_{n \geq 0} I^n$ the Rees ring of $R$ with respect to $I$. A filtration $\{I_n\}_{n \geq 0}$ of ideals of $R$ is called $I$-good if $II_n \subseteq I_{n+1}$ for all $n \geq 0$. For an $I$-good filtration $\{I_n\}_{n \geq 0}$ of $R$ we may form the graded $R(I)$-module $\bigoplus_{n \geq 0} I^n$. Note that it is finitely generated if and only if there is an integer $k$ such that $I_{n+k} \subseteq I^n$ for all $n \geq 1$. Now let $(R, M)$ be a local ring. As the relations

$$(I^n: \langle M \rangle)(I^m: \langle M \rangle) \subseteq I^{n+m}: \langle M \rangle)$$

hold, we may form the graded $R(I)$-algebra $S(I) = \bigoplus_{n \geq 0} I^n: \langle M \rangle$. We are interested in a finiteness criterion of $S(I)$ over $R(I)$. To this end we are looking at the $M$-transform $D_M(\cdot) = \lim_n \text{Hom}_R(M^n, \cdot)$ defined on $R$-modules. If $\text{depth } R > 0$ we have

$$D_M(I^n)D_M(I^m) \subseteq D_M(I^{n+m}) \quad \text{and} \quad I^n: \langle M \rangle \subseteq D_M(I^n),$$

as easily seen. $D_M(I^n)$ is an ideal of $D_M(R)$. Thus we may define the graded $S(I)$-algebra $D(I) = \bigoplus_{n \geq 0} D_M(I^n)$, occurring together with the canonical injection $S(I) \to D(I)$. It follows easily that $D(I) \cong D_{MR(I)}(R(I))$. The next result is the crucial point in order to prove Theorem 2.
PROPOSITION. Let \((R, M)\) denote a local Noetherian ring. For an ideal \(I\) of \(R\) the following conditions are equivalent:

(i) \(D(I)\) is finitely generated over \(R(I)\) and depth \(R > 0\),

(ii) \(S(I)\) is finitely generated over \(R(I)\), and

(iii) \(a(I\hat{R} + p/p) < \dim \hat{R}/p\) for all \(p \in \text{Ass} \hat{R}\).

Proof. The implication (i) \(\Rightarrow\) (ii) holds trivially because of the canonical injection \(S(I) \rightarrow D(I)\). Now let us show (ii) \(\Rightarrow\) (iii). According to (ii) there is an integer \(k\) such that \(I^{n+k} : \langle M \rangle \subseteq I^n\) for all \(n \geq 1\). Let \(p \in \text{Ass} \hat{R}\) and \(p = 0 : x\) for some \(x \in \hat{R}\). By the Artin-Rees lemma there is an integer \(s\) such that \(I^{n+s} \hat{R} \cap x\hat{R} \subseteq xI^n\hat{R}\) for all \(n \geq 1\). Then \(I^{n+k+s}(\hat{R}/p) : \langle M \rangle \subseteq I^n(\hat{R}/p)\) for all \(n \geq 1\) as easily seen. That is, without loss of generality we may assume \(R\) a complete integral domain.

According to D. Rees [10, Theorem 1.4], there is an integer \(t\) such that \((I^n)^{a} \subseteq I^n\) for all \(n \geq 1\). Thus there is an integer \(r\) such that \((I^{n+r})^{a} : \langle M \rangle \subseteq I^n\) for all \(n \geq 1\). Therefore, \(S_a(I) = \bigoplus_{n \geq 0}(I^n)^{a} : \langle M \rangle\) is a finitely generated graded \(R(I)-\)module. Let \(R_a(I) = \bigoplus_{n \geq 0}(I^n)^{a}\). Then it follows that

\[ R(I) \subseteq R_a(I) \subseteq S_a(I) \subseteq R[T], \]

where \(T\) denotes an indeterminate over \(R\). Because \(R_a(I)\) is the integral closure of \(R(I)\) in \(R[T]\) the finiteness of \(S_a(I)\) over \(R(I)\) yields \(R_a(I) = S_a(I)\), i.e., \((I^n)^{a} = (I^n)^{a} : \langle M \rangle\) for all \(n \geq 1\). Hence, \(M \not\subseteq A^a(I)\). Because \(R\) satisfies the altitude formula it follows that \(a(I) < \dim R\) by virtue of S. McAdam [6, Theorem 3]. In order to prove (iii) \(\Rightarrow\) (i) we first note that it is enough to show that \(D(I\hat{R})\) is finitely generated over \(R(I\hat{R})\). Second it is enough to show that \(D(I\hat{R}/p)\) is finitely generated over \(R(I\hat{R}/p)\) for all \(p \in \text{Ass} \hat{R}\) (see M. Brodmann [2, (3.3) and (6.2)]). Without loss of generality we may assume height \(I\hat{R} + p/p\) > 0. Because

\[ a(I\hat{R} + p/p) = \dim R(I\hat{R}/p) \otimes_{\hat{R}} \hat{R}/M, \]

condition (iii) yields height \(MR(I\hat{R}/p) > 1\) according to the altitude formula. Then the statement follows by virtue of M. Brodmann [2, (6.3)].

COROLLARY 2. Let \((R, M)\) be an unmixed local ring. Then the following conditions are equivalent:

(i) There is an integer \(k\) such that \(I^{n+k} : \langle M \rangle \subseteq I^n\) for all \(n \geq 1\).

(ii) \(a(I) < \dim R\).

Proof. According to the Proposition it is enough to show that \(a(I) < \dim R\) is equivalent to condition (iii). Because \(R\) is unmixed we have \(\dim R = \dim \hat{R}/p\) for all \(p \in \text{Ass} \hat{R}\). According to L. J. Ratliff, Jr. [9, (9.2)], we get \(a(I\hat{R} + p/p) \leq a(I)\) for all minimal prime ideals \(p \in \text{Ass} \hat{R}\) and equality holds for some of them.

Corollary 2 answers a question of the author in a preliminary version of [11]. Independently it was shown by D. Katz in [5] and in a private communication.

Proof of Theorem 2. (i) \(\Rightarrow\) (ii): If there is an integer \(k \geq 1\) such that \(P^{(n+k)} \subseteq P^n\) for all \(n \geq 1\), it yields \(P^{(n+k)}R_Q : \langle QR_Q \rangle \subseteq P^{(n+k)}R_Q \subseteq P^nR_Q\) for all \(Q \in A^a(P) \setminus \{P\}\). That is, \(S(PR_Q)\) is finitely generated over \(R(PR_Q)\). Hence the conclusion follows by virtue of the Proposition.
(ii) ⇒ (i): Without loss of generality we may assume \((R, M)\) a local ring as follows by passing to \(R_Q\), \(Q \in A^*(P) \setminus \{P\}\). Now we make an induction on \(\dim R/P\). First let \(\dim R/P = 1\). So it is enough to show that there is an integer \(k\) such that \(P^{n+k} : \langle M \rangle \subseteq P^n\) for all \(n \geq 1\). Hence, the statement follows from the Proposition. Second let \(\dim R/P > 1\). By induction we may assume that there is an integer \(k\) such that \(P^{(n+k)} \subseteq P^n\), \(\langle M \rangle\) for all \(n \geq 1\). Note that the claim is true in any localization \(Q \in A^*(P) \setminus \{P, M\}\). If \(M \notin A^*(P)\), there is nothing to show. Otherwise, it is enough to prove that there is an integer \(r\) with \(P^{n+r} : \langle M \rangle \subseteq P^n\) for all \(n \geq 1\). This is true by virtue of the Proposition.

(ii) ⇒ (iii): Since \(R_g/p\) satisfies the altitude formula it follows that \(Q(R_Q/p) \not\in A^*(P(R_Q/p))\) for all \(Q \in A^*(P) \setminus \{P\}\) and \(p \in \Ass R_Q\) (see S. McAdam [6, Theorem 3]). By virtue of L. J. Ratliff, Jr. [9, (8.3)], we get \(QR_Q \not\in A^*(P(R_Q))\) and \(Q \notin A^*(P)\) for all \(Q \in A^*(P) \setminus \{P\}\). Since \(A^*_a(P) \subseteq A^*(P)\) (see L. J. Ratliff, Jr. [8, (2.6.1)]), we have \(A^*_a(P) = \{P\}\) as required. If \(R\) is locally unmixed all of the above conclusions are reversible, i.e., (iii) ⇒ (ii). □

**Proof of Corollary 1.** It uses the same circle of ideas as given in the proof of Corollary 2, where we have to note that \(\Ass R/(I^n)_a \subseteq \Ass R/(I^{n+1})_a\) (see [8]). □

4. **Proof of Theorem 3.** Before we shall embark in the proof we have to recall a well-known fact. Let \(Q\) denote a prime ideal containing \(P\) and \(S = R \setminus Q\). Then \(\text{gr}_{R_Q}(PR_Q) \equiv \text{gr}_R(P)_{S^*}\), where \(S^*\) denotes the set of leading forms of elements of \(S\) in \(\text{gr}_R(P)\). First assume \(\text{gr}_R(P)_{\text{red}}\) a domain. Since it is stable under localization, it is enough to show \(A^*_a(P) = \{P\}\). By localization we see that it is enough to assume \(R\) is local with maximal ideal \(M\) and to show that it forces \(M \notin A^*_a(P)\). Assume the contrary. Then we may construct an element \(x \in R\) such that \(x^*\), the initial form of \(x\) in \(\text{gr}_R(P)\), is not nilpotent, \(x \notin (P^n)_a\) and \(Mx \subseteq (P^n)_a\) for some \(n\). Choose \(k\) such that \(x \in P^k \setminus P^{k+1}\). Note that \(n \geq k + 1\). For an element \(y \in M \setminus P\) it yields \(xy \in (P^n)_a\), i.e., \((xy)^m + c_1(xy)^{m-1} + \cdots + c_m = 0\) for a certain integer \(m\) and \(c_i \in P^n\). Thus, \((xy)^m \in P^{mk+1}\) and \(x^*y^*\) is nilpotent. Because \(y^*\) is not nilpotent this is a contradiction. In order to prove the converse we first note that \(A^*_a(P) = \{P\}\) yields \(P^{(n)} \subseteq (P^n)_a\) for all \(n \geq 1\), as easily seen. Now suppose there are elements \(x, y\) such that \(x^*y^*\) is nilpotent, while \(y^*\) is not nilpotent. Because \(\text{gr}_{R_P}(PR_P) \equiv \text{gr}_R(P)_{S^*}\), \(S = R \setminus P\), and \(\text{gr}_{R_P}(PR_P)_{\text{red}}\) a domain, the assumption yields that there is an \(s \in R \setminus P\) such that \(s^*x^*\) is nilpotent. Let \(x \in P^1 \setminus P^{i+1}\). Then it follows that \(s^kx^k \in P^{ik+1}\) for a certain \(k \geq 0\). This yields \(x^k \in P^{ik+1} \subseteq (P^{i+1})_a\) and \(x^{km} \in P^{km+1}\) for a certain integer \(m\). Therefore \(x^*\) is nilpotent, as required. □

**Corollary 3.** Let \(P\) be a prime ideal of a commutative Noetherian ring \(R\). Suppose \(\text{gr}_{R_P}(PR_P)\) is a domain (e.g., \(R_P\) is a regular local ring). Then \(\text{gr}_R(P)_{\text{red}}\) is a domain if and only if \(P^{(n)} = (P^n)_a\) for all \(n \geq 1\).

**Proof.** By virtue of Theorem 3 it is enough to show the “only if” part. Because \(A^*_a(P) = \{P\}\) and \(P^{(n)} \subseteq (P^n)_a\), see Theorem 3, we may consider the natural injection \((P^n)_a/P^{(n)} \to R/P^{(n)}\). Therefore it is enough to show \((P^n)_a R_P = P^* R_P\). But this follows easily because \(\text{gr}_{R_P}(PR_P)\) is a domain. □
References


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