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SYMBOLIC POWERS OF PRIME IDEALS AND THEIR TOPOLOGY

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ABSTRACT. For a prime ideal P of a commutative Noetherian ring R a necessary and sufficient condition is given to determine when the P-adic topology is equivalent, resp. linearly equivalent, to the P-symbolic topology. The last means that the symbolic Rees ring is a finitely generated module over the ordinary Rees ring of P. Then it is considered when the integral closure of all the powers of P are primary.

1. Introduction and main results. Let R denote a commutative Noetherian ring. For a prime ideal P of R we define $P^{(n)} = P^n R_P \cap R$, the nth symbolic power of P. Note that $P^{(n)}$ is equal to the uniquely determined P-primary component of P^n. In [3, §7], R. Hartshorne writes: "A general question, whose solution is quite complicated, is to determine when the P-adic topology is equivalent to the P-symbolic topology." Here the P-symbolic topology denotes the topology defined by the symbolic powers $P^{(n)}$, $n \geqslant 1$, of P. In the following we shall give a complete solution to this problem. For an arbitrary ideal I of R note that the sets $\operatorname{Ass} R/I^n$ stabilize for large n (see M. Brodmann [1]). So we denote this stabilized set by $A^*(I)$.

THEOREM 1. *For a prime ideal P of a commutative Noetherian ring R the following conditions are equivalent*:

(i) *The P-adic topology is equivalent to the P-symbolic topology.*

(ii) $\operatorname{height}(P\widehat{R_Q} + p/p) < \dim \widehat{R_Q}/p$ *for all* $Q \in A^*(P) \setminus \{P\}$ *and* $p \in \operatorname{Ass} \widehat{R_Q}$.

Here $\widehat{R_Q}$ denotes the QR_Q-adic completion of R_Q. A particular case of Theorem 1 has been shown by R. Hartshorne [3, Proposition 7.1]. Let P be a prime ideal satisfying condition (i) of the previous theorem. Then, for every integer $n \geqslant 1$ there is an integer $m \geqslant n$ such that $P^{(m)} \subseteq P^n$. Now the question is whether there exists an integer k such that $P^{(n+k)} \subseteq P^n$ for all $n \geqslant 1$. In this case we say that the P-adic topology is linearly equivalent to the P-symbolic topology. It is equivalent to the fact that the "symbolic" Rees ring $\oplus_{n \geqslant 0} P^{(n)}$ is a finitely generated graded module over $\oplus_{n \geqslant 0} P^n$, the ordinary Rees ring of P. For our next result let us recall the notion of analytic spread $a(I)$ of an ideal I of a local Noetherian ring (R, M), i.e., $a(I) = \dim R(I) \otimes_R R/M$, where $R(I) = \oplus_{n \geqslant 0} I^n$ denotes the Rees ring of R with respect to I. Note that height $I \leqslant a(I) \leqslant \dim R$.

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THEOREM 2. *For a prime ideal P of a commutative Noetherian ring R the following conditions are equivalent*:

(i) *The P-adic topology is linearly equivalent to the P-symbolic topology.*

(ii) $a(\widehat{PR_Q} + p/p) < \dim \widehat{R_Q}/p$ *for all* $Q \in A^*(P) \setminus \{P\}$ *and* $p \in \operatorname{Ass} \widehat{R_Q}$.

If one of these conditions is satisfied it follows:

(iii) $A_a^*(P) = \{P\}$.

Moreover, if R is locally unmixed all of the above conditions are equivalent.

Here let I_a denote the integral closure of an ideal I of R, i.e., the set of elements $x \in R$ satisfying an equation $x^m + c_1 x^{m-1} + \cdots + c_m = 0$, $c_i \in I^i$, $i = 1, \ldots, m$. Then $\operatorname{Ass} R/(I^n)_a$ stabilizes for large n, see L. J. Ratliff, Jr. [**8**]. This stabilized set is denoted by $A_a^*(I)$. A local ring (R, M) is called unmixed if $\dim \hat{R}/p = \dim R$ for all $p \in \operatorname{Ass} \hat{R}$, where $\hat{R}$ denotes the M-adic completion. R is called locally unmixed if R_Q is unmixed for all prime ideals Q of R.

COROLLARY 1. *Let P denote a prime ideal of a locally unmixed Noetherian ring R. Then the following conditions are equivalent*:

(i) *There is an integer k such that $P^{(n+k)} \subseteq P^n$ for all $n \geqslant 1$,*

(ii) $a(PR_Q) < \dim R_Q$ *for all* $Q \in A^*(P) \setminus \{P\}$, *and*

(iii) $\operatorname{Ass} R/(P^n)_a = \{P\}$ *for all* $n \geqslant 1$.

The question whether $\operatorname{Ass} R/(P^n)_a = \{P\}$ for all $n \geqslant 1$ is a variation of the problem when does $\operatorname{Ass} R/P^n = \{P\}$ for all $n \geqslant 1$, which has received some previous attention. It is a well-known fact that $\operatorname{gr}_R(P)$ is a domain if and only if $\operatorname{gr}_{R_P}(PR_P)$ is a domain and $\operatorname{Ass} R/P^n = \{P\}$ for all $n \geqslant 1$. Here $\operatorname{gr}_R(I) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$ denotes the form ring of R with respect to an ideal I of R.

THEOREM 3. *Let P denote a prime ideal of a commutative Noetherian ring R. Then* $\operatorname{gr}_R(P)_{\mathrm{red}}$ *is a domain if and only if* $\operatorname{gr}_{R_P}(PR_P)_{\mathrm{red}}$ *is a domain and* $A_a^*(P) = \{P\}$.

If R is a commutative Noetherian ring, by R_{red} we denote the reduced ring of R, i.e., R modulo its nilradical. Hence, $A_a^*(P) = \{P\}$ is, in a certain sense, a transitivity condition. Note that particular cases of Theorem 3 were obtained by C. Huneke [**4**]. Another result related to the question when does $A_a^*(P) = \{P\}$ has been shown by S. McAdam in [**7**].

2. Proof of Theorem 1. Before we shall give the proof, let us fix notation. For two ideals I, J in a Noetherian ring R we define $I : \langle J \rangle = \bigcup_{k \geqslant 1} I : J^k$. Of course $I : \langle J \rangle = I : J^k$ for sufficiently large k.

(i) $\Rightarrow$ (ii): Assume the contrary. Then there are a prime ideal $Q \in A^*(P) \setminus \{P\}$ and a prime ideal $p \in \operatorname{Ass} \widehat{R_Q}$ such that $\dim \widehat{R_Q}/(R\widehat{R_Q} + p) = 0$. By changing the notation we may assume (R, M) a local ring and $\dim \hat{R}/(P\hat{R} + p) = 0$ for some $p \in \operatorname{Ass} \hat{R}$. Then $0 \neq 0 : \langle p \rangle$, because $p \in \operatorname{Ass} \hat{R}$. On the other hand

$$P^n\hat{R} : \langle p \rangle = P^n\hat{R} : \langle P\hat{R} + p \rangle = P^n\hat{R} : \langle M \rangle \subseteq P^{(n)}\hat{R}$$

because $(P\hat{R} + p)^{2k} \subseteq P^k\hat{R} + p^k \subseteq (P\hat{R} + p)^k$ for some integer $k \geqslant 1$ and because $P\hat{R} + p$ is M-primary. Therefore it follows that $0 \neq 0 : \langle p \rangle \subseteq \bigcap_{n \geqslant 1} P^{(n)}\hat{R}$, which is a contradiction to (i).

(ii) $\Rightarrow$ (i): Set $E_{mn} = P^{(m)} + P^n/P^n$ for $m, n \geqslant 1$. Because

$$\operatorname{Ass} E_{mn} \subseteq A^*(P) \setminus \{P\}$$

it is enough to show the statement in any R_Q, $Q \in A^*(P) \setminus \{P\}$. That is, we may assume R a local ring with M its maximal ideal. Now we make an induction on $\dim R/P$. First assume $\dim R/P = 1$. For a fixed integer n, E_{mn} becomes a decreasing sequence of modules of finite length. Hence, it becomes stationary. Set $E_n = P^{(m(n))} + P^n$, where $m(n)$ is an integer such that $E_{mn} = E_{m(n),n}$ for all $m \geqslant m(n)$. It follows that $E_n = E_{n+1} + P^n$ for all n. Now assume $E_k \neq P^k$ for some k. So we may choose elements $x_n \in E_n \setminus P^n$ for all $n \geqslant k$ such that $x_{n+1} \equiv x_n \bmod P^n$. Then $\{x_n\}$ defines a nonzero element of $\varprojlim_n E_n/P^n$. Therefore it follows that

$$0 \neq \bigcap_{n \geqslant 0} E_n = \varprojlim_n E_n/P^n \subseteq \varprojlim_n \hat{R}/P^n\hat{R} \cong \hat{R},$$

because $E_n/P^n \cong E_n/P^n \otimes_R \hat{R}$ is an R-module of finite length. Next we choose an element $x \in \bigcap_{n \geqslant 0} E_n$ such that $0 : x = p$ for some associated prime ideal p of $\hat{R}$. By the Artin-Rees Lemma there exists an integer $r \geqslant k$ such that $P^r\hat{R} \cap x\hat{R} \subseteq xP\hat{R}$. Since $x \in E_r$ and E_r/P^r has finite length, $M^s x \subseteq P^r\hat{R} \cap x\hat{R} \subseteq xP\hat{R}$ for some $s \geqslant 1$. Then $M^s + p \subseteq P\hat{R} + p$, which contradicts (ii). Second suppose $\dim R/P > 1$. As above we consider E_{mn}. As the localizations of these modules in any prime ideal $Q \in A^*(P) \setminus \{P\}$ with $Q \neq M$ tend to zero by induction, they have finite length for m large. Hence, $\operatorname{Ass} E_{mn} \subseteq V(M)$ for m large. Now the proof follows as in the case $\dim R/P = 1$. Hence we omit it. □

REMARK. It follows from the proof of Theorem 1 that we may replace (ii) in Theorem 1 by the following condition:

(ii)′ $\operatorname{height}(P\widehat{R_Q} + p/p) < \dim \widehat{R_Q}/p$ for all $Q \in V(P) \setminus \{P\}$ and $p \in \operatorname{Ass} \widehat{R_Q}$.

Here $V(P)$ denotes the set of prime ideals containing P.

3. Proof of Theorem 2. To this end we have to recall some notation. For an arbitrary ideal I of R we denote by $R(I) = \oplus_{n \geqslant 0} I^n$ the Rees ring of R with respect to I. A filtration $\{I_n\}_{n \geqslant 0}$ of ideals of R is called I-good if $II_n \subseteq I_{n+1}$ for all $n \geqslant 0$. For an I-good filtration $\{I_n\}_{n \geqslant 0}$ of R we may form the graded $R(I)$-module $\oplus_{n \geqslant 0} I_n$. Note that it is finitely generated if and only if there is an integer k such that $I_{n+k} \subseteq I^n$ for all $n \geqslant 1$. Now let (R, M) be a local ring. As the relations

$$(I^n : \langle M \rangle)(I^m : \langle M \rangle) \subseteq I^{n+m} : \langle M \rangle$$

hold, we may form the graded $R(I)$-algebra $S(I) = \oplus_{n \geqslant 0} I^n : \langle M \rangle$. We are interested in a finiteness criterion of $S(I)$ over $R(I)$. To this end we are looking at the M-transform $D_M(\cdot) = \varinjlim_n \operatorname{Hom}_R(M^n, \cdot)$ defined on R-modules. If depth $R > 0$ we have

$$D_M(I^n)D_M(I^m) \subseteq D_M(I^{n+m}) \quad \text{and} \quad I^n : \langle M \rangle \subseteq D_M(I^n),$$

as easily seen. $D_M(I^n)$ is an ideal of $D_M(R)$. Thus we may define the graded $S(I)$-algebra $D(I) = \oplus_{n \geqslant 0} D_M(I^n)$, occuring together with the canonical injection $S(I) \to D(I)$. It follows easily that $D(I) \cong D_{MR(I)}(R(I))$. The next result is the crucial point in order to prove Theorem 2.

PROPOSITION. *Let (R, M) denote a local Noetherian ring. For an ideal I of R the following conditions are equivalent*:

(i) *$D(I)$ is finitely generated over $R(I)$ and* depth $R > 0$,

(ii) *$S(I)$ is finitely generated over $R(I)$, and*

(iii) *$a(I\hat{R} + p/p) < \dim \hat{R}/p$ for all $p \in \operatorname{Ass} \hat{R}$.*

PROOF. The implication (i) $\Rightarrow$ (ii) holds trivially because of the canonical injection $S(I) \to D(I)$. Now let us show (ii) $\Rightarrow$ (iii). According to (ii) there is an integer k such that $I^{n+k}\colon \langle M\rangle \subseteq I^n$ for all $n \geqslant 1$. Let $p \in \operatorname{Ass} \hat{R}$ and $p = 0 : x$ for some $x \in \hat{R}$. By the Artin-Rees lemma there is an integer s such that $I^{n+s}\hat{R} \cap x\hat{R} \subseteq xI^n\hat{R}$ for all $n \geqslant 1$. Then $I^{n+k+s}(\hat{R}/p)\colon \langle M\rangle \subseteq I^n(\hat{R}/p)$ for all $n \geqslant 1$ as easily seen. That is, without loss of generality we may assume R a complete integral domain. According to D. Rees [**10**, Theorem 1.4], there is an integer t such that $(I^{n+t})_a \subseteq I^n$ for all $n \geqslant 1$. Thus there is an integer r such that $(I^{n+r})_a\colon \langle M\rangle \subseteq I^n$ for all $n \geqslant 1$. Therefore, $S_a(I) = \oplus_{n\geqslant 0}(I^n)_a\colon \langle M\rangle$ is a finitely generated graded $R(I)$-module. Let $R_a(I) = \oplus_{n\geqslant 0}(I^n)_a$. Then it follows that

$$R(I) \subseteq R_a(I) \subseteq S_a(I) \subseteq R[T],$$

where T denotes an indeterminate over R. Because $R_a(I)$ is the integral closure of $R(I)$ in $R[T]$ the finiteness of $S_a(I)$ over $R(I)$ yields $R_a(I) = S_a(I)$, i.e., $(I^n)_a = (I^n)_a : \langle M\rangle$ for all $n \geqslant 1$. Hence, $M \notin A_a^*(I)$. Because R satisfies the altitude formula it follows that $a(I) < \dim R$ by virtue of S. McAdam [**6**, Theorem 3]. In order to prove (iii) $\Rightarrow$ (i) we first note that it is enough to show that $D(I\hat{R})$ is finitely generated over $R(I\hat{R})$. Second it is enough to show that $D(I(\hat{R}/p))$ is finitely generated over $R(I(\hat{R}/p))$ for all $p \in \operatorname{Ass} \hat{R}$ (see M. Brodmann [**2**, (3.3) and (6.2)]). Without loss of generality we may assume height$(I\hat{R} + p/p) > 0$. Because

$$a(I\hat{R} + p/p) = \dim R(I(\hat{R}/p)) \otimes_{\hat{R}} \hat{R}/M,$$

condition (iii) yields height $MR(I(\hat{R}/p)) > 1$ according to the altitude formula. Then the statement follows by virtue of M. Brodmann [**2**, (6.3)]. □

COROLLARY 2. *Let (R, M) be an unmixed local ring. Then the following conditions are equivalent*:

(i) *There is an integer k such that $I^{n+k}\colon \langle M\rangle \subseteq I^n$ for all $n \geqslant 1$.*

(ii) *$a(I) < \dim R$.*

PROOF. According to the Proposition it is enough to show that $a(I) < \dim R$ is equivalent to condition (iii). Because R is unmixed we have $\dim R = \dim \hat{R}/p$ for all $p \in \operatorname{Ass} \hat{R}$. According to L. J. Ratliff, Jr. [**9**, (9.2)], we get $a(I\hat{R} + p/p) \leqslant a(I)$ for all minimal prime ideals $p \in \operatorname{Ass} \hat{R}$ and equality holds for some of them. □

Corollary 2 answers a question of the author in a preliminary version of [**11**]. Independently it was shown by D. Katz in [**5**] and in a private communication.

PROOF OF THEOREM 2. (i) $\Rightarrow$ (ii): If there is an integer $k \geqslant 1$ such that $P^{(n+k)} \subseteq P^n$ for all $n \geqslant 1$, it yields $P^{n+k}R_Q\colon \langle QR_Q\rangle \subseteq P^{(n+k)}R_Q \subseteq P^nR_Q$ for all $Q \in A^*(P)\setminus\{P\}$. That is, $S(PR_Q)$ is finitely generated over $R(PR_Q)$. Hence the conclusion follows by virtue of the Proposition.

(ii) $\Rightarrow$ (i): Without loss of generality we may assume (R, M) a local ring as follows by passing to R_Q, $Q \in A^*(P) \setminus \{P\}$. Now we make an induction on $\dim R/P$. First let $\dim R/P = 1$. So it is enough to show that there is an integer k such that P^{n+k}: $\langle M \rangle \subseteq P^n$ for all $n \geqslant 1$. Hence, the statement follows from the Proposition. Second let $\dim R/P > 1$. By induction we may assume that there is an integer k such that $P^{(n+k)} \subseteq P^n$: $\langle M \rangle$ for all $n \geqslant 1$. Note that the claim is true in any localization $Q \in A^*(P) \setminus \{P, M\}$. If $M \notin A^*(P)$, there is nothing to show. Otherwise, it is enough to prove that there is an integer r with P^{n+r}: $\langle M \rangle \subseteq P^n$ for all $n \geqslant 1$. This is true by virtue of the Proposition.

(ii) $\Rightarrow$ (iii): Since $\widehat{R_Q/p}$ satisfies the altitude formula it follows that $Q(\widehat{R_Q/p}) \notin A_a^*(P(\widehat{R_Q/p}))$ for all $Q \in A^*(P) \setminus \{P\}$ and $p \in \operatorname{Ass} \widehat{R_Q}$ (see S. McAdam [**6**, Theorem 3]). By virtue of L. J. Ratliff, Jr. [**9**, (8.3)], we get $Q\widehat{R_Q} \notin A_a^*(P\widehat{R_Q})$ and $Q \notin A_a^*(P)$ for all $Q \in A^*(P) \setminus \{P\}$. Since $A_a^*(P) \subseteq A^*(P)$ (see L. J. Ratliff, Jr. [**8**, (2.6.1)]), we have $A_a^*(P) = \{P\}$ as required. If R is locally unmixed all of the above conclusions are reversible, i.e., (iii) $\Rightarrow$ (ii). $\square$

PROOF OF COROLLARY 1. It uses the same circle of ideas as given in the proof of Corollary 2, where we have to note that $\operatorname{Ass} R/(I^n)_a \subseteq \operatorname{Ass} R/(I^{n+1})_a$ (see [**8**]). $\square$

4. Proof of Theorem 3. Before we shall embark in the proof we have to recall a well-known fact. Let Q denote a prime ideal containing P and $S = R \setminus Q$. Then $\operatorname{gr}_{R_Q}(PR_Q) \cong \operatorname{gr}_R(P)_{S^*}$, where S^* denotes the set of leading forms of elements of S in $\operatorname{gr}_R(P)$. First assume $\operatorname{gr}_R(P)_{\text{red}}$ a domain. Since it is stable under localization, it is enough to show $A_a^*(P) = \{P\}$. By localization we see that it is enough to assume R is local with maximal ideal M and to show that it forces $M \notin A_a^*(P)$. Assume the contrary. Then we may construct an element $x \in R$ such that x^*, the initial form of x in $\operatorname{gr}_R(P)$, is not nilpotent, $x \notin (P^n)_a$ and $Mx \subseteq (P^n)_a$ for some n. Choose k such that $x \in P^k \setminus P^{k+1}$. Note that $n \geqslant k+1$. For an element $y \in M \setminus P$ it yields $xy \in (P^n)_a$, i.e., $(xy)^m + c_1(xy)^{m-1} + \cdots + c_m = 0$ for a certain integer m and $c_i \in P^{ni}$. Thus, $(xy)^m \in P^{mk+1}$ and x^*y^* is nilpotent. Because y^* is not nilpotent this is a contradiction. In order to prove the converse we first note that $A_a^*(P) = \{P\}$ yields $P^{(n)} \subseteq (P^n)_a$ for all $n \geqslant 1$, as easily seen. Now suppose there are elements x, y such that x^*y^* is nilpotent, while y^* is not nilpotent. Because $\operatorname{gr}_{R_P}(PR_P) \cong \operatorname{gr}_R(P)_{S^*}$, $S = R \setminus P$, and $\operatorname{gr}_{R_P}(PR_P)_{\text{red}}$ is a domain, the assumption yields that there is an $s \in R \setminus P$ such that s^*x^* is nilpotent. Let $x \in P^t \setminus P^{t+1}$. Then it follows that $s^k x^k \in P^{tk+1}$ for a certain $k \geqslant 0$. This yields $x^k \in P^{(tk+1)} \subseteq (P^{tk+1})_a$ and $x^{km} \in P^{tkm+1}$ for a certain integer m. Therefore x^* is nilpotent, as required. $\square$

COROLLARY 3. *Let P be a prime ideal of a commutative Noetherian ring R. Suppose $\operatorname{gr}_{R_P}(PR_P)$ is a domain (e.g., R_P is a regular local ring). Then $\operatorname{gr}_R(P)_{\text{red}}$ is a domain if and only if $P^{(n)} = (P^n)_a$ for all $n \geqslant 1$.*

PROOF. By virtue of Theorem 3 it is enough to show the "only if" part. Because $A_a^*(P) = \{P\}$ and $P^{(n)} \subseteq (P^n)_a$, see Theorem 3, we may consider the natural injection $(P^n)_a/P^{(n)} \to R/P^{(n)}$. Therefore it is enough to show $(P^n)_a R_P = P^n R_P$. But this follows easily because $\operatorname{gr}_{R_P}(PR_P)$ is a domain. $\square$

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