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A FORMULA FOR AN EQUIVALENT INVARIANT MEASURE

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ABSTRACT. Let T be a nonsingular 1-1 measurable transformation of a finite measure space $(X, \mathscr{B}, m)$. A simple construction is given for a σ-finite measure μ, equivalent to m and invariant under T, when such a measure exists.

1. Introduction. In [**1**] the author gave necessary and sufficient conditions for the existence of a σ-finite invariant measure μ for a nonsingular 1-1 measurable transformation T of a measure space $(X, \mathscr{B}, m)$ onto itself, where μ is required to be equivalent to m, and where (without real loss of generality but with considerable simplification of details) m is assumed finite. My first proof that the conditions are sufficient was long and complicated; the much better proof in [**1**] (kindly supplied by Y. Ito) is indirect. Here we give an even simpler proof, which provides a simple explicit formula for such an invariant measure (which, of course, need not be unique in general). We recall the following notation from [**1**].

The Radon-Nikodym derivative of T^n $(n \in \mathbf{N} = \{0,1,2,\ldots\})$ is ω_n, so $m(T^nB) = \int_B \omega_n(x)\,dm(x)$ for all $B \in \mathscr{B}$. We may (and do) freely discard invariant null sets from X, and thus arrange that $0 < \omega_n(x) < \infty$ for all $x \in X$ and $n \in \mathbf{N}$. For all $\alpha, \beta \geqslant 0$, the "relative density" $d(\alpha, \beta, x)$, here renamed $\delta(\alpha, \beta, x)$, is

$$\lim_{k\to\infty} \frac{|\{n \in \mathbf{N} : n \leqslant k \text{ and } \omega_n(x) \geqslant \alpha\}|}{|\{n \in \mathbf{N} : n \leqslant k \text{ and } \omega_n(x) \geqslant \beta\}|};$$

this limit exists (possibly infinite) for all $\alpha, \beta \geqslant 0$ and (almost) all x. We put

$$I_\beta(x) = \int_0^\infty \delta(\alpha, \beta, x)\,d\alpha,$$

and have that $\beta \leqslant I_\beta(x) \leqslant \infty$; $I_\beta(x)$ increases with β (for fixed x); and (an observation I owe to A. H. Stone)

$$\int_0^\infty \frac{d\beta}{I_\beta(x)} = 0 \text{ or } 1 \quad \text{for (almost) all } x.$$

(See [**1**, p. 226].)

To avoid repetition the term "invariant measure" in what follows means "σ-finite measure, equivalent to m, and invariant under T".

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The main theorem of [1] asserts that an invariant measure exists if, and only if,

$$\int_0^\infty \frac{d\beta}{I_\beta(x)} = 1 \quad \text{a.e.} \tag{1}$$

(For some other equivalent conditions, see [1, Theorem 4].)

2. The proof of "only if" in [1] is relatively straightforward; here we re-prove the converse by deriving a simple formula for a suitable measure, assuming that (1) holds.

First we note that (whether or not (1) holds) we have $\omega_n(x) = \omega(x)\omega_{n-1}(Tx)$ (where $\omega(x)$ is written for $\omega_1(x)$), and thus

$$\begin{aligned}\delta(\alpha, \beta, x) &= \lim_{k\to\infty} \frac{|\{n \in \mathbf{N} : n \leqslant k \text{ and } \omega_{n-1}(Tx) \geqslant \alpha/\omega(x)\}|}{|\{n \in \mathbf{N} : n \leqslant k \text{ and } \omega_{n-1}(Tx) \geqslant \beta/\omega(x)\}|} \\ &= \delta(\alpha/\omega(x), \beta/\omega(x), Tx),\end{aligned}$$

whence a simple calculation gives

$$I_\beta(x) = \omega(x) I_{\beta/\omega(x)}(Tx),$$

and so, if $b \geqslant 0$,

$$\int_0^b \frac{d\beta}{I_\beta(x)} = \int_0^{b/\omega(x)} \frac{d\beta}{I_\beta(Tx)}. \tag{2}$$

Now assume (1), and fix x for the present. If $I_\beta(x)$ becomes infinite for some $\beta > 0$, put β^* $(= \beta^*(x)) = \inf\{\beta\colon I_\beta(x) = \infty\}$; then the function, defined for $b \geqslant 0$ by

$$b \mapsto \int_0^b \frac{d\beta}{I_\beta(x)};$$

is continuous, nondecreasing, and strictly increasing for $0 \leqslant b \leqslant \beta^*$, and takes the value 1 at $b = \beta^*$. Thus there is a unique $b(x) > 0$ such that

$$\int_0^{b(x)} \frac{d\beta}{I_\beta(x)} = \frac{1}{2}. \tag{3}$$

If $I_\beta(x)$ remains finite for all (finite) $\beta > 0$, similar, but simpler, reasoning again gives a unique $b(x) > 0$ satisfying (3).

Now from (2) we have

$$\int_0^{b(x)/\omega(x)} \frac{d\beta}{I_\beta(Tx)} = \frac{1}{2},$$

so

$$b(x)/\omega(x) = b(Tx). \tag{4}$$

Also, $b(x)$ is a measurable function of x, for $\{x\colon b(x) > c\}$ is precisely the set of x's for which $\int_0^c d\beta/I_\beta(x) < 1/2$; and this is measurable for each c, by Fubini's theorem.

Thus (4) (by a familiar and elementary calculation) shows that the measure μ defined by

$$\mu(B) = \int_B b(x)\, dm(x) \qquad (B \in \mathscr{B})$$

is T-invariant, as required.

3. Remarks. (i) Even if (1) does not hold, the set of x's for which $\int_0^\infty d\beta/I_\beta(x) = 1$ is the largest subset of X (modulo null sets) on which an invariant measure exists, and the present construction, restricted to this subset, gives an invariant measure on it.

(ii) When (1) holds we can obtain *all* (σ-finite, m-equivalent) T-invariant measures by replacing the constant value $1/2$ in the definition of $b(x)$ by an arbitrary measurable invariant function f with values in $(0, 1)$.

(iii) Necessary and sufficient conditions for the existence of a *finite* invariant measure, and simple formulae for one when it exists, are well known; see for example [**1**, Theorem 1].

References

1. Dorothy Maharam, *Invariant measures and Radon-Nikodym derivatives*, Trans. Amer. Math. Soc. **135** (1969), 223–248.

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