TOPOLOGICAL EQUIVALENCE IN THE SPACE OF INTEGRABLE VECTOR-VALUED FUNCTIONS

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Abstract. The Banach space \( L^1(0, T; X) \) is retopologized by \( \| f \| = \max_{0 \leq a < b \leq T} \int_a^b \| f(t) \| \, dt \), where \( \| \cdot \| \) is the norm in the given Banach space \( X \). It is shown here that this topology coincides with the usual weak topology of \( L^1(0, T; X) \) on a wide class of weakly compact subsets.

Let \( X \) be a Banach space and \( T > 0 \). Denote by \( L^1(0, T; X) \) the Banach space of all (Bochner) integrable functions (equivalence classes) on \([0, T] \) with the norm

\[
\| f \|_1 = \int_0^T \| f(\tau) \| \, d\tau,
\]

where \( \| \cdot \| \) is the norm in \( X \). \( L^1(0, T; X) \) can be retopologized with the weaker norm

\[
\| f \| = \max_{0 \leq a < b \leq T} \left\| \int_a^b f(\tau) \, d\tau \right\|.
\]

Denote this space by \( L^1(\| \cdot \|) \). Note that

\[
\| f \|_1 = \max_{0 \leq a < T} \left\| \int_a^T f(\tau) \, d\tau \right\|
\]

is an equivalent norm in \( L^1(\| \cdot \|) \). The space \( L^1(\| \cdot \|) \) was recently used to obtain existence results for some kinds of abstract differential equations (see e.g. [4], [5]). It was observed that subsets of the form

\[
\{ f \in L^1(0, T; X) : f(t) \in K \text{ almost everywhere on } [0, T] \}
\]

are compact in \( L^1(\| \cdot \|) \) if the set \( K \) is compact in \( X \). The main purpose of this note is to show that for a wide class of subsets of \( L^1(0, T; X) \) (particularly of the above type), the topology generated by \( \| \cdot \| \) coincides with the usual weak topology of \( L^1(0, T; X) \). On the connections between \( L^1(0, T; H) \) and \( L^1(\| \cdot \|) \), where \( H \) is a Hilbert space, see [5].

Definition. We say that a set \( F \subset L^1(0, T; X) \) has property (U) if:

(i) \( F \) is bounded and uniformly integrable.

(ii) For every \( \varepsilon > 0 \) there exists a compact set \( K_\varepsilon \subset X \) such that for every \( f \in F \) there exists a measurable set \( \Omega_{f,\varepsilon} \) with \( \mu([0, T] \setminus \Omega_{f,\varepsilon}) \leq \varepsilon \) and \( f(t) \in K_\varepsilon \) for \( t \in \Omega_{f,\varepsilon} \).

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Here \( \mu \) is the Lebesgue measure on \([0, T]\). On the property (U), see [1 and 3].

Recall that if we denote by \( l_1 \) the Banach space of all real summable sequences \( \{x_n\}^\infty_{n=1} \) with norm \( \|x\|_1 = \sum_{n=1}^{\infty} |x_n|_1 \), then the conjugate space \( m \) is the space of all real bounded sequences \( \{y_n\}^\infty_{n=1} \) with norm \( \|y\|_\infty = \sup_{n} |y_n|_1 \) where \( y = \{y_n\}^\infty_{n=1} \in m \). Let \( K \) be a compact set. Denote by \( C(K) \) the Banach space of all continuous functions on \( K \) with sup-norm \( |\cdot|_\infty \). The conjugate \((C(K))^* = M(K)\) is the Banach space of all Radon measures on \( K \). The conjugate \((L^1(0, T; X))^* \) is the space \( \Lambda(0, T; X^*) \) of all essentially bounded scalar measurable functions \( g : [0, T] \to X^* \), and every linear continuous functional on \( L^1(0, T; X) \) is given by

\[
\int_0^T (f(\tau), g(\tau)) \, d\tau,
\]

where \( f \in L^1(0, T; X) \), \( g \in \Lambda(0, T; X^*) \) and \((\cdot, \cdot)\) is the pairing between \( X \) and \( X^* \).

(See [2, 8.14–8.18].) Our main result is the following:

**Theorem.** Let \( X \) be a Banach space and \( T > 0 \). Let the set \( F \subseteq L^1(0, T; X) \) have property (U). Then the weak topology of \( L^1(0, T; X) \) and the topology of \( L^1(||\cdot||) \) coincide on \( F \). Moreover, \( F \) is relatively compact in \( L^1(||\cdot||) \).

**Proof.** Note that the set \( \bigcup_{k=1}^{\infty} K_{1/n} \) is separable in \( X \) (\( K_{1/n} \) is defined as in the Definition). Therefore \( F \) is separable in \( L^1(0, T; X) \) and we can suppose without loss of generality, that \( X \) is separable.

Let \( \{t_n\}^\infty_{n=1} \) be a dense sequence in \([0, T]\) and \( \{y^*_n\}^\infty_{n=1} \) a weak-star dense sequence in the unit ball \( S^* \) of \( X^* \). Let \( \chi[t_n, t_m] \) be a characteristic function of the interval \([t_n, t_m]\). Then the set of all the functions of the form \( y^*_n \cdot \chi[t_p, t_q], \, t_p < t_q, \) is countable. Denote these functions by \( \{\phi^*_n\}^\infty_{n=1} \). Then

\[
\|f\| = \sup_n \left| \int_0^T (f(\tau), \phi^*_n(\tau)) \, d\tau \right|.
\]

Define a linear continuous operator \( P : L^1(0, T; X) \to m \) by

\[
Pf = \left( \int_0^T (f(\tau), \phi^*_n(\tau)) \, d\tau \right)^\infty_{n=1}.
\]

Note that \( \|f\| = |Pf|_\infty \).

We will prove that \( P(F) \subseteq m \) is relatively (norm) compact in \( m \). Suppose there exists a compact set \( K \subseteq X \) such that \( F \subseteq F(K) = \{ f \in L^1(0, T; X) : f(\tau) \in K \) almost everywhere on \([0, T]\)\}. Recall that \( C(K) \) is the Banach space of continuous functions on \( K \), and define the operator \( \hat{P} : l_1 \to L^1(0, T; C(K)) \) by \( \hat{P}e_n = \phi^*_n \) on the standard basis \( \{e_n\}^\infty_{n=1} \) of \( l_1 \), and then extend it by linearity and continuity to all \( l_1 \). The set \( \{\phi^*_n\}^\infty_{n=1} \subseteq L^1(0, T; C(K)) \) is norm compact. This can be checked directly or by using the criterion of compactness in the spaces \( L^p(0, T; X) \) (see [3, Theorem A.1]). Thus \( \hat{P} \) is compact. Its dual, \( \hat{P}^* : (L^1(0, T; C(K)))^* \to m \), is also compact. Thus \( \hat{P}^* : \Lambda(0, T; M(K)) \to m \),

\[
\hat{P}^*g = \left( \int_0^T (\phi^*_n(\tau), g(\tau)) \, d\tau \right)^\infty_{n=1}.
\]
where the pairing $(\cdot, \cdot)$ is
\[
(\phi_n(\tau), g(\tau)) = \int_K \phi_n(x, \tau) \, dg(x, \tau)
\]
for the measure $dg(x, \tau)$ corresponding to $g(\tau)$. In particular, if $g(\tau)$ is a Dirac measure for each $\tau \in [0, T]$ and $g(\tau)$ is concentrated at $f(\tau) \in K \subset X$, then $g \in \Lambda(0, T; M(K))$ and
\[
\hat{P}g = \left\{ \int_0^T \left( f(\tau), \phi_n(\tau) \right) \, d\tau \right\}_{n=1}^\infty = Pf.
\]
Thus the action of the operator $\hat{P}$ on $F$ can be identified with the action of the operator $\hat{P}^*$, and the image $P(F)$ is relatively compact in $m$. Now we can suppose that $F$ is a general set with property (U).

Consider the sets $F_\varepsilon = \{ f \cdot \chi(\Omega_{f,\varepsilon}) : f \in F \}$. Here $\Omega_{f,\varepsilon}$ is a measurable set as in the Definition and $\chi(\Omega_{f,\varepsilon})$ is its characteristic function. The set $F$ is uniformly integrable, hence for each $\delta > 0$ there exists an $\varepsilon > 0$ such that $\inf\{ |f - g|_1 : g \in F_\varepsilon \} < \delta$ for every $f \in F$. Note that $||h|| \leq |h|_1$ for each $h \in L^1(0, T; X)$. Therefore, by definition of $P$, we have $\inf\{ |Pf - y|_1 : y \in PF_\varepsilon \} \leq \delta$. But any set $PF_\varepsilon$ is relatively compact in $m$, hence the set $PF \subset m$ is relatively compact. Since $P: L^1(0, T; X) \to m$ is continuous in the norm topologies, it is also continuous in the weak topologies. The weak and strong topologies coincide on $PF$. Thus the restriction $P|_F$ is continuous if we take the weak topology in $L^1(0, T; X)$ and the strong one in $m$. By [1, Proposition 13] any set $F$ with property (U) is relatively weakly compact in $L^1(0, T; X)$. Since we can suppose that $F$ is convex, the linear map $P: F \to PF$ is a homeomorphism in these topologies. The strong topology of $m$ on $PF$ is the strong topology of $L^1(||\cdot||)$ on $F$. Thus $F$ is relatively compact in $L^1(||\cdot||)$ and the theorem is proved.

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References


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