MARKOV'S INEQUALITY FOR POLYNOMIALS WITH REAL ZEROS

PETER BORWEIN

Abstract. Markov's inequality asserts that \( \| p_n' \| \leq n^2 \| p_n \| \) for any polynomial \( p_n \) of degree \( n \). (We denote the supremum norm on \([-1, 1]\) by \( \| \cdot \| \).) In the case that \( p_n \) has all real roots, none of which lie in \([-1, 1]\), Erdős has shown that \( \| p_n' \| \leq c n \| p_n \| /2 \). We show that if \( p_n \) has \( n - k \) real roots, none of which lie in \([-1, 1]\), then \( \| p_n' \| \leq c n (k + 1) \| p_n \| \), where \( c \) is independent of \( n \) and \( k \). This extension of Markov's and Erdős' inequalities was conjectured by Szabados.

Introduction. Markov's inequality asserts that

\[
\| p_n' \|_{[-1,1]} \leq n^2 \| p_n \|_{[-1,1]}
\]

for any polynomial \( p_n \in \pi_n \) [2 and 3]. (\( \pi_n \) denotes the algebraic polynomials of degree at most \( n \) and \( \| \cdot \|_{A} \) denotes the supremum norm on \( A \).) Erdős [1] in 1940 offered the following refinement of Markov's inequality. If \( p_n \in \pi_n \) and \( p_n \) has all its roots in \( \mathbb{R} - (-1, 1) \), then

\[
\| p_n' \|_{[-1,1]} \leq \frac{c n}{2} \| p_n \|_{[-1,1]}. \tag{2}
\]

Inequality (1) iterates to give bounds for the \( k \)th derivative of a polynomial. However, we cannot proceed inductively with inequality (2) since some of the roots of the derivatives may be in \([-1, 1]\). With this in mind, Szabados and Varma established a version of (2) for polynomials of degree \( n \) with all real roots and at most one root in \([-1, 1]\), namely, for such a polynomial \( p_n \),

\[
\| p_n' \|_{[-1,1]} \leq c_1 n \| p_n \|_{[-1,1]}, \tag{3}
\]

where \( c_1 \) is independent of \( n \) [5]. This, of course, yields the following inequality:

\[
\| p_n^{(k)} \|_{[-1,1]} \leq c_2 n^k \| p_n \|_{[-1,1]} \tag{4}
\]

for any \( p_n \in \pi_n \) that has all its roots in \( \mathbb{R} - (-1, 1) \). In [6] Szabados proposed the following

Conjecture. If \( p_n \) is a polynomial of degree \( n \) and \( p_n \) has at least \( n - k \) roots in \( \mathbb{R} - (-1, 1) \), then there is a constant \( c \) (\( c \leq 9 \)) so that

\[
\| p_n' \|_{[-1,1]} \leq c n (k + 1) \| p_n \|_{[-1,1]} \tag{5}
\]
It is our intention to prove this slightly strengthened form of Szabados' conjecture. In its original form the conjecture had the additional assumption that all the roots of $p_n$ be real. Up to the constant this result is best possible; Szabados in [6] constructs polynomials $p_n$ of degree $n$ with $n - k$ roots in $\mathbb{R} - (-1, 1)$ so that

$$\|p_n\|_{[-1,1]} \geq \frac{n \cdot k}{2} \|p_n\|_{[-1,1]} \quad (0 < k \leq n).$$

It is apparent from (1) and (2) that the best constant must depend on $k$. Some related results may be found in [4].

Inequalities for the higher derivatives of polynomials with real roots can now be derived straightforwardly from (5). For example,

**Theorem.** If $p_n \in \pi_n$ has at least $n - k$ zeros in $\mathbb{R} - (-1, 1)$, then

$$\|p_n^{(m)}\|_{[-1,1]} \leq c_m (n - m)! k! \|p_n\|_{[-1,1]},$$

where $c_m \leq 9^m$ depends only on $m$.

**2. Proof of the Conjecture.** Let $c_{2k}$ be the $2k$th Chebychev polynomial shifted to the interval $[0, 2]$ and normalized to have lead coefficient 1. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ be the roots of $c_{2k}$ in $[1, 2]$ and let

$$t_k := \prod_{i=1}^{k} (x - \alpha_i).$$

**Lemma 1.** The polynomial $q_k := (x + 2m/k)^{m+k} t_k(x)$ has the following property. If $\alpha_0 = 0$ and $\alpha_{n+1} = 1$, then, for $i = 1, 2, \ldots, n$,

$$\left\| \left( x + \frac{2m}{k} \right)^{m+k} t_k \right\|_{[\alpha_i, \alpha_{i+1}]} > \left\| \left( x + \frac{2m}{k} \right)^{m+k} t_k \right\|_{[\alpha_i, \alpha_{n+1]}},$$

where the maximums on successive intervals occur with alternating sign.

**Proof.** Let $0 < \beta_1 < \cdots < \beta_k$ be the roots of $c_{2k}$ in $[0, 1]$. We observe that $x/(x - \beta_i)$ is positive and decreasing on $[\beta_k, \infty]$ and that $c_{2k}$ equioscillates on the intervals in question (i.e. $c_{2k}$ satisfies (7) with equality). We now note that

$$x^k t_k = \left( \prod_{i=1}^{k} \frac{x}{(x - \beta_i)} \right) c_{2k}$$

satisfies the conclusion of Lemma 1. To finish the proof we need only observe that $(x + 2m/k)^{m+k}/x^k$ is decreasing on $[0, 2]$. □

Let $n = 2k + m$ and let

$$s_n(x) := \frac{1}{(1 + m/k)^n} q_k \left( \left( 1 + \frac{m}{k} \right) x + \left( 1 - \frac{m}{k} \right) \right).$$

(We have shifted from $[-2m/k, 2]$ to $[-1, 1]$.) This polynomial will act as a kind of near extremal polynomial for the Conjecture. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ be the roots of $s_n$ in $(-1, 1)$. We collect the properties of $s_n$ that we require in the next lemma.
Lemma 2. For \( n = m + 2k \) and \( s_n \) as above:

(a) \( s_n(x) = (x + 1)^{m+k} \prod_{i=1}^{k}(x - \gamma_i) \),

(b) \( \sum_{i=1}^{k} \left( \frac{1}{1 - \gamma_i} \right) \leq 4k(n - k) \), and

(c) for \( i = 1, \ldots, k \), \( \gamma_0 = -1 \) and \( \gamma_{k+1} = 1 \)

\[ \|s_n\|_{\gamma_{i-1}, \gamma_i} \geq \|s_n\|_{\gamma_i, \gamma_{i+1}}. \]

Proof. Parts (a) and (c) are immediate from the construction of \( s_n \). Part (b) follows from the observation that, for \( \alpha_i \) as in (6).

\[ \sum_{i=1}^{k} \frac{1}{2 - \alpha_i} \leq \frac{c_{2k}(2)}{c_{2k}(2)} = 4k^2 \]

and the observation that

\[ 1 - \gamma_i = \frac{1}{1 + m/k} (2 - \alpha_i). \]

Let \( p^*_n \in \pi_n \) maximize

\[ |p_n^*(1)|/\|p_n\|_{[-1,1]}, \]

where the maximum is taken over all polynomials in \( \pi_n \) that have all but at most \( k \) roots in \( \mathbb{R} \setminus (-1,1) \). The information we need about \( p^*_n \) is contained in the next lemma.

Lemma 3. Let \( p^*_n \) be as above. Then

(a) \( p^*_n \) has \( k \) simple roots \( \delta_1 < \cdots < \delta_k \) in \((-1,1)\), \( p^*_n \) has \( n - k \) roots at \( \pm 1 \), and \( p^*_n \) achieves its maximum modulus on each of the intervals \([-1, \delta_1], [\delta_1, \delta_2], \ldots, [\delta_k, 1]\).

(b) Either \( p^*_n \) has no roots in \([ -1, \infty ) \) or \( p^*_n \) has exactly one root at 1.

Proof. The proof of (a) is a simple and standard perturbation argument (if \( p^*_n \) did not satisfy (a) then it would be possible to perturb \( p^*_n \) to reduce its norm on \([ -1,1 ]\) without decreasing the derivative at 1). We will prove only that \( p_n \) has no roots in \((1, \infty)\), the other parts are similar. First suppose that \( p^*_n \) has two roots at \( \alpha > 1 \) and \( \beta > 1 \). Consider

\[ v_n(x) := \frac{p_n^*(x)(x - 1)^2}{(x - \alpha)(x - \beta)}. \]

Then for sufficiently small \( \epsilon > 0 \):

(i) \( \|p_n^* - \epsilon v_n\|_{[-1,1]} < \|p_n^*\|_{[-1,1]} \),

(ii) \( |(p_n^* - \epsilon v_n)'(1)| = |p_n^*(1)| \),

(iii) \( p_n^* - \epsilon v_n \) has all but at most \( k \) roots in \( \mathbb{R} \setminus (1,1) \).

Part (iii) follows since \((x - \alpha)(x - \beta) - \epsilon(x - 1)^2 \) has two roots in \([1, \infty)\) for sufficiently small \( \epsilon \). (Note \( \alpha \) may equal \( \beta \).) However, this contradicts the maximality of \( p_n^* \).

Next we suppose that \( p^*_n \) has exactly one (nonrepeat) root at \( \alpha > 1 \). Now we argue as before by considering

\[ v_n(x) := \frac{p_n^*(x)(x - 1)^2}{(x - 1)(x - \alpha)}. \]
If \( p_\ast'(1) \neq 0 \) we must observe that in this case \( \text{sign}(p_\ast'(1)) = \text{sign}(p_n^\ast'(1)) \) and, hence, that

\[
\nu_\ast'(1) = \frac{(1 - \alpha) p_n^\ast(1)}{(1 - \alpha)^2}
\]

has the opposite sign to \( p_n^\ast(1) \). The last observation requires noticing that if \( p_n^\ast'(1) \) has opposite sign to \( p_n^\ast(1) \), then \( p_n^\ast' \) has all its zeros in \(( -\infty, 1] \) and, hence, \( p_n''(1) \neq 0 \). Thus, \( |p_n^\ast'| \) is increasing on \([1, \infty)\), \( |p_n| \) is decreasing on \([1, \alpha)\) and \( p_n^\ast(x + (\alpha - 1)) \) violates the maximality assumptions on \([-1, 1]\).

Part (b) follows since if \( p_n^\ast \) has two or more zeros at 1, then \( p_n^\ast(1) \) would also equal zero.

**Lemma 4.** If \( p_n \in \pi_n \) has at least \((n - k)\) roots in \( \mathbb{R} - (-1, 1) \), then

\[
|p_n^\ast(1)| < \frac{n}{2}(k + 1)n\|p_n\|_{[-1,1]}.
\]

**Proof.** If \( 2k < n \), then the lemma follows from Markov’s inequality, so we may suppose \( 2k > n \). Suppose there exists \( p_n \), as above, so that

\[
|p_n^\ast(1)| < \frac{n}{2}(k + 1)(n - k)\|p_n\|_{[-1,1]},
\]

and let \( q_n \) be the maximal such \( p_n \). By Lemma 3, this \( q_n \) equioscillates \( k + 1 \) times on \([1, \infty)\). We shall first consider the case where \( q_n \) has no root at 1.

The key to the proof is to observe that the roots of \( q_n \) lie to the left of the roots of \( s_n \) (as defined in Lemma 3). We may write

\[
q_n(x) = (x + 1)^{n-k} \prod_{i=1}^{k} (x - \rho_i),
\]

where \(-1 < \rho_1 < \cdots < \rho_k < 1\). Also,

\[
s_n(x) = (x + 1)^{n-k} \prod_{i=1}^{k} (x - \gamma_i).
\]

The claim is that \( \gamma_i > \rho_i \) for each \( i \). This is seen as follows. Choose the largest \( i \) for which \( \rho_i > \gamma_i \). Then pick \( \eta \) so that \( \|\eta q_n\|_{[\gamma_i,1]} = \|s_n\|_{[\gamma_i,1]} \). (We will specify the sign of \( \eta \) later.) We can deduce from the equioscillation of \( q_n \) that \( \eta q_n - s_n \) has at least \( k - i \) roots on \([\beta, 1]\), where \( \beta \) is the first point greater than \( \rho_i \), where \( \eta q_n \) achieves its maximum modulus. From Lemma 2(c) we deduce that \( \eta q_n - s_n \) has at least \( i - 1 \) roots on \((-1, \alpha)\), where \( \alpha \) is the largest point less than \( \gamma_i \) where \( s_n \) achieves its maximum modulus. We need only observe that if we choose the sign of \( \eta \) so that

\[
\text{sign} \eta q_n(\beta) = -\text{sign} s_n(\alpha),
\]

then \( \eta q_n - s_n \) must have 2 roots in \((\alpha, \beta)\). Thus, \( \eta q_n - s_n \) has \( n + 1 \) roots which is a contradiction and we conclude that \( \rho_i \leq \gamma_i \).
We now observe that, since $\rho_i \leq \gamma_i < 1$,
\[
\frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}} = \frac{q_n'(1)}{q_n(1)} = \sum_{i=1}^{k} \frac{1}{1 - \rho_i} + \sum_{i=1}^{n-k} \frac{1}{1 - (-1)} 
\leq \sum_{i=1}^{k} \frac{1}{1 - \gamma_i} + \frac{n - k}{2} \leq 4k(n - k) + \frac{n - k}{2},
\]
where the later inequality follows from Lemma 2. This is a contradiction.

In the case where $q_n$ has exactly one root at 1 we proceed as follows. Let $d > 1$ be the unique point in $(1, \infty)$, where $|q_n'(d)| = \|q_n\|_{[-1,1]}$. We can now consider $q_n$ on $[-1, d]$. We note that
\[
\frac{|q_n'(d)|}{\|q_n\|_{[-1,1]}} \geq \frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}}
\]
since $|q_n'|$ is increasing on $[1, \infty)$. We can repeat verbatim the argument of the first part applied to
\[d = \frac{d + 1}{2} + \left(\frac{d - 1}{2}\right)\]
with $k$ replaced by $k + 1$. This allows us to deduce the contradiction that
\[
\frac{|q'(1)|}{\|q_n\|_{[-1,1]}} \leq \frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}} \leq 4(k + 1)(n - k) + \frac{n - k}{2}.
\]

The proof of the Conjecture is now straightforward.

**Proof of Conjecture.** Let $p_n$ be a polynomial of degree $n$ with $n - k$ roots in $\mathbb{R} - (-1, 1)$. Let $x_0$ be a point in $[-1, 1]$, where $p_n'$ achieves its maximum modulus. We suppose $x_0 \leq 0$ ($x_0 > 0$ follows analogously). Let $ax + b$ map $[x_0, 1]$ one-to-one onto $[-1, 1]$ in such a way that $x_0 \to 1$.

Note that $|a| < 2$. Thus, if $v_n(ax + b) = p_n(x)$, then
\[
\frac{|p_n'(x_0)|}{\|p_n\|_{[-1,1]}} \leq \frac{2|v'(1)|}{\|v_n\|_{[-1,1]}} = 9n(k + 1),
\]
where the last inequality follows from Lemma 4 and the observation that $v_n$ has at least as many roots as $p_n$ in $\mathbb{R} - [-1, 1]$. □

**References**


Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia B3H 4H8, Canada