ITERATES OF HOLOMORPHIC SELF-MAPS
OF THE UNIT BALL IN HILBERT SPACE

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ABSTRACT. An example is given of a biholomorphic self-mapping $T$ of the
unit ball in infinite-dimensional Hilbert space satisfying $0 = \lim \inf_{n} \|T^n(0)\| < \lim \sup_{n} \|T^n(0)\| = 1$. 

1. Introduction. Let $B$ denote an open unit ball in the complex Hilbert space $H$ and let $T$ be a holomorphic self-mapping of the ball $B$ with no fixed points in $B$. The sequence of iterates of such a mapping was considered by several authors. In case $H = \mathbb{C}$ (the complex plane) the basic result comes from the old papers of J. Wolff [15] and A. Denjoy [3] (see also [1]). They showed that the iterates converge to a unimodular constant (uniformly on compact subsets of the unit disc $\Delta$). Recently it was generalized [11, 13] to self-mappings of the unit ball $B \subset \mathbb{C}^n$ with $n > 1$.

In the infinite-dimensional case analogous results have been obtained in several special cases only [7, 14]. It was not quite clear whether or not a similar statement holds for all holomorphic, fixed point free mappings $T : B \to B$. Our aim in this paper is to construct an example showing that the answer is negative even for automorphisms (i.e. biholomorphic self-mappings of the ball $B$).

Recall that $B$ can be furnished with the invariant hyperbolic metric $\rho$ so that every holomorphic mapping $T : B \to B$ becomes $\rho$-nonexpansive [5, 10]. Recently it has been shown that there exist interesting analogies between properties of $\rho$-nonexpansive self-mappings of the ball $B$ and of norm-nonexpansive mappings on the whole space $H$ [6, 9, 12]. Therefore our question is closely related to a similar problem concerning a norm-nonexpansive, fixed point free mapping $F : H \to H$. For such a mapping one can show that if $\dim H < \infty$, then $\lim_{n \to \infty} \|F^n(x)\| = \infty$ for all $x$ in $H$ (see [2 and 16] for even more general results). On the other hand M. Edelstein [4] gave an example of an isometry $F : l^2 \to l^2$ satisfying the following conditions:

(i) $F$ has no fixed point in $l^2$;
(ii) $\{F^n(0)\}$ is norm-unbounded;
(iii) There exists a subsequence $\{F^{n_k}(0)\}$ with $\lim_{k \to \infty} F^{n_k}(0) = 0$
(see [4] for details). Here $l^2$ denotes the space of all sequences $x = (x_n)$ of complex numbers with the usual Hilbert norm. Let us notice that $F$ can be easily rewritten in the form

(iv) $F(x) = f(x) + a$,

where $x \in H$, $a = F(0)$ and $f$ is a linear isometry onto $l^2$. 

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Our construction below is based upon this example.

2. Construction of the example. Let the infinite-dimensional complex Hilbert space $H$ be given. Recall first that the unit ball $B \subset H$ is the biholomorphic image of the region (the Siegel upper half-space) $\Omega = \{ (\lambda, w) \in \mathbb{C} \times H : \text{Im} \lambda > ||w||^2 \}$ under the Cayley transform $C$ (see, for instance, [8, Chapter 2, §31]). Notice that $C$ maps the point $(i, 0) \in \Omega$ into the origin. Clearly every holomorphic mapping $T : B \to B$ can be obtained from a suitable holomorphic mapping $\Phi : \Omega \to \Omega$ by composition on the right and left by $C^{-1}$ and $C$ respectively.

Now let $K$ be a real Hilbert space such that $H = K \oplus iK$. Denote by $F_1$ an isometry of the space $K$ which satisfies properties (i)-(iv) with $l^2$ replaced by $K$ (the existence of such an isometry follows immediately from Edelstein's example). Clearly $F_1$ can be extended to an isometry $F : H \to H$ also satisfying (i)-(iv) and, in addition, satisfying

(v) $F(K) = K$.

Now we define the mapping $\Phi : \Omega \to \Omega$ by

$$\Phi(\lambda, w) = [\lambda + i ||a||^2 + 2i \langle f(w), a \rangle, f(w)]$$

($\langle . , . \rangle$ denotes the inner product in $H$). It is easy to observe that $\Phi$ is an automorphism of $\Omega$. A little calculation shows that (1) can be rewritten in the form

$$\Phi(\lambda, w) = [\lambda + i(||F(w)||^2 - ||w||^2) - 2 \text{Im} \langle f(w), a \rangle, F(w)],$$

which reduces for $w \in K$ to

$$\Phi(\lambda, w) = [\lambda + i(||F(w)||^2 - ||w||^2), F(w)].$$

Hence we obtain

$$\Phi^n(\lambda, w) = [\lambda + i(||F^n(w)||^2 - ||w||^2), F^n(w)]$$

for every $w \in K$, $\text{Im} \lambda > ||w||^2$ and $n = 1, 2, \ldots$

Set $(\lambda, w) = (i, 0)$ in (2). We have

$$\Phi^n(i, 0) = [i + i ||F^n(0)||^2, F^n(0)].$$

From (ii) and (iii) it immediately follows that $\Phi^n(i, 0) \to (i, 0)$ as $k \to \infty$ and that $\{\Phi^n(i, 0)\}$ is unbounded in $\Omega$.

Put now $T = C \circ \Phi \circ C^{-1}$. It is easy to observe that $T$ is a fixed point free automorphism of the ball $B$ such that $\limsup_n ||T^n(0)|| = 1$ and $T^n(0) \to 0$ as $k \to \infty$. Thus $T$ is the desired example.

REFERENCES


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