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A SHARP INEQUALITY FOR THE p-CENTER OF GRAVITY OF A RANDOM VARIABLE

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ABSTRACT. Let X be a real-valued random variable. For $p > 1$, define the p-center of gravity of X, $C_p(X)$, as the unique number c which minimizes $\|X - c\|_p$. This paper exhibits a more-or-less explicit expression for the best constant $\gamma = \gamma_p$ in the inequality $|E(X) - C_p(X)| \leqslant \gamma\|X - C_p(X)\|_p$, and presents asymptotic formulas for γ_p as $p \to 1, 2$ and $+\infty$, respectively. The definition of $C_p(X)$ is extended to variables taking values in an arbitrary Hilbert space H, and it is shown that γ_p is not increased by this extension.

1. Introduction. Let X be a real-valued random variable. For $p \geqslant 1$, define the p-center of gravity of X, $C_p(X)$, as the number c which minimizes $\|X - c\|_p$. One has $C_1(X) =$ any median of X, while $C_2(X) = E(X)$. In general, $C_p(X)$ is uniquely defined for $p > 1$, since the function $c \to \|X - c\|_p$ is then strictly convex. A question which immediately arises is: How far can $C_p(X)$ be from $E(X)$? §2 of this paper exhibits a more-or-less explicit expression for

$$\gamma_p = \sup\{|E(X)|: C_p(X) = 0, \|X\|_p = 1\} \qquad (p > 1).$$

In general, one has

$$|E(X) - C_p(X)| \leqslant \gamma_p\|X - C_p(X)\|_p \leqslant \gamma_p\|X\|_p.$$

The best constant in the related inequality $\|X - E(X)\|_p \leqslant D_p\|X\|_p$ was determined by Witsenhausen [5]. His results for D_p can be obtained, and in some cases extended, using methods very similar to those of the present paper; see [1] for details. Numerical values for γ_p are provided, as well as asymptotic formulas valid for $p \to 1$, $p \to 2$, and $p \to \infty$, respectively. In §3, the definition of $C_p(X)$ is extended to random variables X taking values in an arbitrary Hilbert space H. It is shown that the constant γ_p is not increased by this extension.

2. Main results.

THEOREM 1. *Let $p > 1, p \neq 2$, and $q = p/(p-1)$. One has*

$$(1) \quad \gamma_p = \sup_{x>0} \frac{x - x^{q-1}}{(x^q + x)^{1/p}(1+x)^{1/q}} = \sup_s \frac{|\operatorname{Sinh}(q-2)s|}{[\operatorname{Cosh}(q-1)s]^{1/p}[\operatorname{Cosh} s]^{1/q}},$$

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from which it follows that $\gamma_p = \gamma_q$. *The second supremum in* (1) *is attained exactly for* $s = \pm s_p$, *where* s_p *is the unique positive root of the equation*

$$\left\{(q-1)^2 \operatorname{Tanh}(q-1)s + \operatorname{Tanh} s\right\} \operatorname{Tanh}(q-2)s = q(q-2). \tag{2}$$

PROOF. From the obvious symmetry of the problem, one has

$$\gamma_p = \sup\left\{E(X) \colon C_p(X) = 0,\ E\left(|X|^p\right) = 1\right\}.$$

The condition $C_p(X) = 0$ is equivalent to $E((\operatorname{sgn} X)|X|^{p-1}) = 0$. Hence, from a well-known principle of the theory of moments (see, e.g., Theorem 3 of [**3**]), one has

$$\gamma_p = \inf\left\{A + C \colon \text{there exists } B \text{ such that } x \leqslant A|x|^p + B(\operatorname{sgn} x)|x|^{p-1} + C\right\}.$$

From Theorems 4 and 5 of [**3**], one may confine attention, in evaluating γ_p, to random variables supported by the associated contact sets $\{t\colon\ t = A|t|^p + B(\operatorname{sgn} t)|t|^{p-1} + C\}$. Now, the inequality $x \leqslant A|x|^p + B(\operatorname{sgn} x)|x|^{p-1} + C$ is equivalent to

$$mx + b \leqslant \phi_d(x) \overset{\text{def}}{\equiv} |x|^p + d(\operatorname{sgn} x)|x|^{p-1}, \tag{3}$$

where $m = A^{-1} > 0$, $b = -A^{-1}C$ and $d = A^{-1}B$. Inequality (3) holds iff the line $y = mx + b$ supports the convex hull of the graph of ϕ_d from below. The graph of ϕ_0 is convex. For $d \neq 0$, the graph of ϕ_d is first convex, then concave and finally convex again. It follows that each contact set $\{t\colon\ t = A|t|^p + B(\operatorname{sgn} t)|t|^{p-1} + C\}$, mentioned above, consists of at most two points. Thus, in determining γ_p one may assume that X takes at most two values (one positive and one negative, since $E((\operatorname{sgn} X)|X|^{p-1}) = 0$).

Let $u, v > 0$ with $P(X = v) = a$, $P(X = -u) = 1 - a$. The conditions $E((\operatorname{sgn} X)|X|^{p-1}) = 0$ and $E(|X|^p) = 1$ lead to the equations $av^{p-1} - (1-a)u^{p-1} = 0$ and $av^p + (1-a)u^p = 1$, respectively. Further, $E(X) = av - (1-a)u$. Introducing $x = a/(1-a) = (u/v)^{p-1}$, one obtains

$$E(X) = f(x) \overset{\text{def}}{\equiv} \frac{x - x^{q-1}}{(x^q + x)^{1/p}(1+x)^{1/q}},$$

which proves the first formula in (1). The second follows on substituting $s = (\ln x)/2$. Interchanging p and q, and replacing s by $s/(q-1)$, one finds $\gamma_p = \gamma_q$. Setting to zero the logarithmic derivative of the second formula in (1) gives the equation

$$(q-2)\operatorname{Coth}(q-2)s = \frac{q-1}{p}\operatorname{Tanh}(q-1)s + \frac{1}{q}\operatorname{Tanh} s,$$

which is equivalent to (2). The monotonicity of the function Tanh implies that (2) has a unique positive root s_p, at which the second supremum in (1) is attained.

In some cases, (2) can be solved explicitly. For example, if $p = 3$ one has $\gamma_3 = \gamma_{3/2}$, so letting $q = 3$ in (2), one finds the equation

$$(4\operatorname{Tanh} 2s + \operatorname{Tanh} s)\operatorname{Tanh} s = 3,$$

leading to $\gamma_3 = \gamma_{3/2} = (2\sqrt{3} - 3)^{1/2}/4^{1/3}$. Likewise, $\gamma_4 = \gamma_{4/3} = 2^{1/2}3^{-3/4}$. In general, (2) is easily solved numerically. Table I provides a sample of values of s_p and γ_p. For values of p near 1, 2 or $+\infty$, asymptotic formulas for s_p and γ_p can be obtained.

TABLE I. Selected Values of s_p and γ_p

p	s_p	γ_p
2.1	1.257757391	0.063081196
2.2	1.312362783	0.120239909
2.3	1.363924906	0.172125389
2.4	1.412794038	0.219321410
2.5	1.459259873	0.262346991
3.0	1.662885890	0.429160632
4.0	1.975436846	0.620403240
5.0	2.213433877	0.722673691
10.0	2.940014234	0.891257102
20.0	3.655311	0.954265
30.0	4.06997	0.97163
40.0	4.36279	0.97957
50.0	4.58924	0.98408

THEOREM 2. *For $p \to 2$, one has*

$$s_p = s_* + s_*(p-2)/2 - s_*(p-2)^2/6 + O(p-2)^3, \tag{4}$$

where s_* is the unique positive root of the equation $s \operatorname{Tanh} s = 1$, and

$$\gamma_p = (s_*^2 - 1)^{1/2}|p-2|\left\{1 - \frac{1}{2}(p-2) + \left(\frac{1}{8} + \frac{s_*^2}{24}\right)(p-2)^2\right\} + O(p-2)^4. \tag{5}$$

PROOF. It is clear from (2) that s_p remains bounded away from both 0 and $+\infty$ as $p \to 2$, equivalently, $q \to 2$. For example, suppose a sequence $p_i \to 2$ can be found with $s_{p_i} \to \infty$. Then, $[\operatorname{Tanh}(q_i - 2)s_{p_i}]/(q_i - 2) \to \infty$, a contradiction. Thus, for $q \to 2$, (2) may be written $s \operatorname{Tanh} s = 1 + O(q-2)$. It follows that $s_p = s_* + O(q-2)$. Standard resubstitution techniques (see, e.g., [**4**, p. 11]) lead to (4), after some calculation. Equation (5) follows routinely from (4).

THEOREM 3. *For $p \to \infty$, equivalently, $q \downarrow 1$, one has*

$$s_p = \ln(2p-2) + \frac{(\ln(2p-2))^2}{2(p-1)^2} + \frac{1}{4(p-1)^2} + O\left(\frac{\ln p}{p}\right)^3 \tag{6}$$

and

$$\gamma_p = 1 - \frac{\ln 2}{p-1} - \frac{\ln(p-1)}{(p-1)^2} - \frac{1-(\ln 2)^2}{2(p-1)^2} + O\left(\frac{\ln p}{p}\right)^3. \tag{7}$$

PROOF. Setting $q = 1$ in (2) given, $\operatorname{Tanh}^2 s = 1$. Thus, $s_p \to \infty$ as $p \to \infty$. Let $\varepsilon = q - 1$. Then (2) becomes

$$\varepsilon^2 \operatorname{Tanh} \varepsilon s + \operatorname{Tanh} s \operatorname{Tanh}(1 - \varepsilon)s = 1 - \varepsilon^2. \tag{8}$$

For $x \to \infty$, $\operatorname{Tanh} x = 1 - 2e^{-2x} + O(e^{-4x})$. I claim that $\varepsilon s_p \to 0$ as $p \to \infty$. For, suppose a sequence $p_i \to \infty$ can be found with $\varepsilon_i s_{p_i} \geqslant c > 0$ for all i. Since $\operatorname{Tanh} s_{p_i}$ and $\operatorname{Tanh}(1 - \varepsilon_i)s_{p_i}$ are then equal to $1 + o(\varepsilon^2)$, and $\operatorname{Tanh} c > 0$, it follows that the left side of (8) eventually exceeds 1, which is a contradiction. Equation (8) may now be written $\operatorname{Tanh}^2 s = 1 - \varepsilon^2 + o(\varepsilon^2)$, so that

$$S = \ln\left(\frac{2}{\varepsilon}\right) + o\left(\ln\left(\frac{2}{\varepsilon}\right)\right) = \ln(2p - 2) + o(\ln(2p - 2)).$$

Continuing in this manner one arrives at formulas (6) and (7) by routine, though tedious, calculation.

Finally, note that (6) and (7) also describe the behavior of s_p and γ_p, respectively, as $p \to 1$, since $s_q = (q - 1)s_p$ and $\gamma_q = \gamma_p$.

3. Extensions to vector-valued random variables. First consider the complex case. Let Z be a complex-valued random variable with $\|Z\|_p = E(|Z|^p)^{1/p} < \infty$, where $p > 1$. Because $|z|^p$ is a strictly convex function on $\mathbf{C}$, $C_p(Z)$ may be uniquely defined by $\|Z - C_p(Z)\|_p = \inf_{c \in \mathbf{C}} \|Z - c\|_p$. Let $\gamma_p(\mathbf{C}) = \sup\{|E(Z)|: \|Z\|_p = 1, C_p(Z) = 0\}$.

THEOREM 4. $\gamma_p(\mathbf{C}) = \gamma_p$.

PROOF. The condition $C_p(Z) = 0$ is equivalent to $E(\operatorname{Re} Z|Z|^{p-2}) = E(\operatorname{Im} Z|Z|^{p-2}) = 0$. Replacing Z by $e^{i\theta}Z$, if necessary, one may assume $E(Z)$ is real, i.e., $E(\operatorname{Im} Z) = 0$. Hence, in determining $\gamma_p(\mathbf{C})$, inequalities of the following type must be investigated [3]:

$$\operatorname{Re} z \leqslant A + B \operatorname{Re} z|z|^{p-2} + C \operatorname{Im} z|z|^{p-2} + D \operatorname{Im} z + F|z|^p,$$

where A, B, C, D and F are real constants. On replacing z by $\bar{z}$, one may take $C = D = 0$ and so focus attention on inequalities of the form

$$\operatorname{Re} z \leqslant \tilde{A} + \tilde{B}|z|^p + \tilde{C} \operatorname{Re} z|z|^{p-2}, \tag{9}$$

valid for all $z \in \mathbf{C}$, with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ as real constants. Let $z = re^{i\theta}$ and fix r. Then (9) takes one of the two forms $\operatorname{Cos} \theta \leqslant d$ or $\operatorname{Cos} \theta \geqslant d$, for $0 \leqslant \theta < 2\pi$, where d is real. To check these inequalities only the values $\theta = 0, \pi$ need be considered. It follows that (1) holds for all complex z if it holds for all real z. Since (9) for z real reduces to $x \leqslant \tilde{A} + \tilde{B}|x|^p + \tilde{C}(\operatorname{sgn} x)|x|^{p-1}$, the theorem is proved.

Next, let H be an arbitrary real Hilbert space and $X \in L_p(H)$. It is well known that $L_p(H)$ is a uniformly convex Banach space; a proof may be based on Lemma 15.4 and Theorem 15.7 of [2]. Hence, if S is a closed linear subspace of $L_p(H)$, and $x \notin S$, there is a unique $s_0 \in S$ with $\|s_0 - x\| = \inf_{s \in S} \|s - x\|$. In particular, $C_p(X)$ may be defined as the unique element of H which minimizes $\|X - c\|_p$. Now let $\gamma_p(H) = \sup\{\|E(X)\|: \|X\|_p = 1, C_p(X) = 0\}$.

THEOREM 5. $\gamma_p(H) = \gamma_p$.

PROOF. Clearly, $\gamma_p(H) \geqslant \gamma_p$, since $\mathbf{R} \subseteq H$. Since $\gamma_p = \gamma_p(\mathbf{C})$, one has, from (9), that there exist real constants $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ with $\tilde{A} + \tilde{B} = \gamma_p$ and

$$x \leqslant \tilde{A} + \tilde{B}(x^2 + y^2)^{p/2} + \tilde{C}x(x^2 + y^2)^{(p/2)-1} \tag{10}$$

for all $x, y \in \mathbf{R}$. Choose an orthonormal basis (x_α) for H with $x_{\alpha_0} = E(X)/\|E(X)\|$. From (10),

$$(x, x_{\alpha_0}) \leqslant A + B\|x\|^p + C(x, x_{\alpha_0})\|x\|^{p-2} \tag{11}$$

for all $x \in H$. Now suppose $\|X\|_p = 1$ and $C_p(X) = 0$. The latter implies, in particular, that

$$\inf_{t \in \mathbf{R}} E\left\{\left[(X, x_{\alpha_0}) - t\right]^2 + \sum_{\alpha \neq \alpha_0} (X, x_\alpha)^2\right\}^{p/2}$$

is assumed at $t = 0$. By differentiation, $E(X, x_{\alpha_0})\|X\|^{p-2} = 0$. Morever, $E(X, x_{\alpha_0}) = \|E(X)\|$. From (11), therefore, $\|E(X)\| \leqslant A + B = \gamma_p$. Thus, $\gamma_p(H) \leqslant \gamma_p$ and so $\gamma_p(H) = \gamma_p$.

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