A CHARACTERIZATION OF CLARKE'S STRICT TANGENT CONE VIA NONLINEAR SEMIGROUPS

JEAN - PAUL PENOT

Abstract. Clarke's strict tangent cone $T^s_f(a)$ at $a \in X$ to a closed subset of a Banach space $E$ is shown to contain the limit inferior of tangent cones $T_x(x)$ to $X$ at $x$ as $x \to a$, $x \in X$. Several characterizations of $T^s_f(a)$ are presented. As a consequence various tangential and subtangential conditions for continuous vector fields on $X$ are shown to be equivalent.

It is well known that invariance results for dynamical systems on closed subsets of Banach spaces are linked with tangency conditions (see, for instance, [4-7, 10, 12, 16, 17] and their references). On the other hand, some results on tangent cones can be deduced from the study of dynamical systems (see, for instance, [12, 16]). In this note we follow this second streamline in order to give characterizations of the strict tangent cone (or Clarke's tangent cone) to a closed subset in a Banach space. These characterizations were given in [13] under restrictive assumptions, valid for instance in the finite dimensional case. Other studies containing some of these implications can be found in [1, 4, 7-9] for instance.

1. Characterizations of the strict tangent cone. Given a subset $X$ of a Banach space $E$ we denote by $d_X(e)$ the distance of $e \in E$ to $X$: $d_X(e) = \inf \{d(e, x) : x \in X\}$. We set $B(a, r) = \{x \in X : d(a, x) \leq r\}$.

Let us recall that the classical tangent cone (also called contingent cone) at $a \in X$ to $X$ is the set $T_X(a)$ (also denoted elsewhere $T(X, a)$ or $T^c_a$) of vectors $v \in E$ such that

$$d_X(a, v) = \limsup_{(t, e) \to (0+, a)} t^{-1}(d_X(e + tv) - d_X(e)).$$

The strict tangent cone (or Clarke's tangent cone) is the set $T^s_X(a)$ of vectors $v \in E$ such that $d^s_X(a, v) \leq 0$, where

$$d^s_X(a, v) = \liminf_{(t, w) \to (0+, v)} t^{-1}(d_X(a + tw) - d_X(a)).$$

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Both cones have been extensively used in optimization and nonlinear analysis. The first one gives a closer approximation to the set $X$ around $a$ than the second one; but it is not necessarily convex. The second one does not necessarily increase with $X$. Some algebraic links between the two cones have been delineated in [11 and 14]. Here we focus our attention on the relationships between the two cones obtained by taking limits as $x \to a$, $x \in X$.

The following result was shown in [2] to be a consequence of a general ordering principle (see also [6] for related material).

**Proposition 1.** Let $F$ be a closed subset of a Banach space $E$, let $c \in \mathbb{R}_+$, $\omega \in ]0, +\infty[$ and let $S$ be a continuous semigroup on $E$ such that

(a) $d(S(t)x, S(t)y) \leq e^{\omega t}d(x, y)$ for each $(t, x, y) \in \mathbb{R}_+ \times E \times E$,

(b) $\liminf_{t \to 0} t^{-1}d(S(t)x, F) \leq c$ for each $x \in F$.

Then $d(S(t)z, F) \leq e^{\omega t}d(z, F) + c\omega^{-1}(e^{\omega t} - 1)$ for each $t \in \mathbb{R}_+$ and each $z \in E$.

In the following theorem the restrictive assumptions made in [13] are dropped.

**Theorem 1.** Let $a$ be a point of a closed subset $X$ of a Banach space $E$. For any $v \in E$ the following assertions are equivalent:

(a) $v \in T^+_a(X)$;

(b) $\limsup_{t \to 0} \sup_{x \in X} t^{-1}(d_X(e + tv) - d_X(e)) \leq 0$;

(b') $\lim_{x \to a, x \in X} \sup_{t \to 0} t^{-1}d_X(x + tv) = 0$;

(c) $\lim_{x \to a, x \in X} \inf_{t \to 0} t^{-1}(d_X(e + tv) - d_X(e)) \leq 0$;

(c') $\lim_{x \to a, x \in X} \inf_{t \to 0} t^{-1}d_X(x + tv) = 0$.

**Proof.** The implications (b) $\Rightarrow$ (b'), (c) $\Rightarrow$ (c'), (b) $\Rightarrow$ (c) and (b') $\Rightarrow$ (c') are obvious; the implication (a) $\Rightarrow$ (b) follows from the following inequality in which $q(t, e) = t^{-1}(d_X(e + tv) - d_X(e))$:

$$\inf_{\alpha > 0} \sup_{e \in B(a, \alpha)} \sup_{\beta > 0 \in ]0, \beta[} \inf_{t \to 0} q(t, e).$$

Thus it suffices to show that (c') implies (a). There is no loss of generality in supposing $\|v\| \leq 1$. Let $\varepsilon > 0$ be given; we will show that

$$\limsup_{(t, e) \to (0_+ , a)} t^{-1}(d_X(e + tv) - d_X(e)) \leq 3\varepsilon.$$

Using (c') we can find $\delta > 0$ such that $\liminf_{t \to 0} t^{-1}d_X(x + tv) \leq \varepsilon$ for each $x \in X \cap B$, where $B = B(a, 3\delta)$ is the closed ball with center $a$ and radius $3\delta$. Let $F = X \cap B$, let $\lambda(e) = \delta^{-1}\min(\delta, d(e, B^c))$ with $B^c = E \setminus B$, and let $S$ be the semigroup generated by the lipschitzian vector field $V: E \to E$ given by $V(e) = \lambda(e)v$. As is well known, the flow of $V$ is defined on $\mathbb{R} \times E$ as $V$ is globally lipschitzian (with Lipschitz constant $\omega = \delta^{-1}$) so that $S$ is well defined and satisfies condition (a) of Proposition 1 (cf. [3] for instance). Let us check condition (b): for each $x \in F$

$$t^{-1}d(S(t)x, F) \leq t^{-1}d(S(t)x, x + tV(x)) + t^{-1}d(x + tV(x), F).$$
As \( \lim_{t \to 0^+} t^{-1}(S(t)x - x) = V(x) \), the first term of the right-hand side has limit 0; the second one has limit inferior \( \lambda(x) \lim_{s \to 0^+, s \to 0^+} d(x + sv, X) \leq \epsilon \) when \( d(a, x) < 3\delta, x \in X \) and limit 0 when \( d(a, x) = 3\delta, x \in X \). Thus condition (b) is satisfied with \( c = \epsilon \).

For each \( z \in B(a, \delta) \) we have \( d_X(z) \leq d(z, a) \leq \delta \), and any \( x \in X \) such that \( d(z, x) \leq 2\delta \) is in \( B(a, 3\delta) \), so that \( d_X(z) = d(z, F) \). On the other hand, as \( F \subset X \) we have \( d_X(z + tw) \leq d(z + tw, F) \) for any \( t \in \mathbb{R}_+ \). Let \( \alpha \in ]0, \delta[ \) be so small that \( 2\alpha \leq \epsilon \), \( t^{-1}(e^{\omega t} - 1) \leq 2\omega \) for \( t \in ]0, \alpha[ \). Then for \( t \in ]0, \alpha[ \), \( z \in B(a, \alpha) \) we have \( S(t)z = z + tw \) and

\[
t^{-1}(d_X(z + tw) - d_X(z)) \leq t^{-1}(d(z + tw, F) - d(z, F)) \leq t^{-1}(e^{\omega t} - 1)(d(z, F) + \omega^{-1}\epsilon) \leq 2\omega(\alpha + \omega^{-1}\epsilon) \leq 3\epsilon.
\]

2. Some consequences. Given \( x \in X \) and \( v \in E \) we define the contingency coefficient (or tangency coefficient) of \( v \) at \( x \) with respect to \( X \) as

\[
k_X(x, v) = \lim_{t \to 0^+} t^{-1}d_X(x + tv).
\]

Let us set

\[
T^*_X(x) = \{ v \in E : k_X(x, v) \leq \epsilon \|v\| \},
\]

so that \( T^*_X(x) \) is a cone and \( T_X(x) = \bigcap_{\epsilon > 0} T^*_X(x) \).

**Corollary 1.** For any closed subset \( X \) of a Banach space and any \( a \in X \) one has

\[
\liminf_{x \to a, x \in X} \lim_{\epsilon \to 0^+} T^*_X(x) \subset T^*_X(a).
\]

**Proof.** Suppose \( v \) belongs to the left-hand side of this inclusion. Then for any \( \alpha > 0 \) there exists \( \beta > 0 \) such that for any \( x \in X \cap B(a, \beta) \) and any \( \epsilon \in ]0, \beta[ \) there exists \( v' \in T^*_X(x) \cap B(v, \alpha) \). Thus

\[
k_X(x, v) \leq k_X(x, v') + d(v', v) \leq \epsilon \|v\| + \alpha.
\]

As \( \epsilon \in ]0, \beta[ \) is arbitrary, we get \( k_X(x, v) \leq \alpha \) for \( x \in X \cap B(a, \beta) \) whence \( \lim_{x \to a, x \in X} k_X(x, v) = 0 \) and \( v \in T^*_X(a) \) by Theorem 1. \( \square \)

**Corollary 2.** For any closed subset \( X \) of a Banach space and any \( a \in X \) one has

\[
\liminf_{x \to a, x \in X} T_X(x) \subset T^*_X(a).
\]

This follows from the preceding corollary and the fact that \( T_X(x) \subset T^*_X(x) \) for any \( x \in X \) and any \( \epsilon > 0 \).

**Corollary 3.** Let \( X \) be a closed subset of a Banach space \( E \) and let \( a \in E \). Suppose \( v \in E \) is such that for any subset \( A \) of \( X \) with \( a \) in its closure one has \( v \in \limsup_{x \to a, x \in A} T_X(x) \). Then \( v \in T^*_X(a) \).
This follows from the fact that for any relation \( F: X \to E \) one has
\[
\liminf_{x \to a} F(x) = \bigcap_{A \in \mathcal{A}} \limsup_{x \to a} F(x),
\]
where \( \mathcal{A} \) is the family of subsets \( A \) of \( X \) whose closure contains \( a \).

**Corollary 4.** Let \( V: X \to E \) be a continuous vector field on a closed subset \( X \) of a Banach space \( E \). Then the following assertions are equivalent:

(a) for each \( x \in X \), \( V(x) \in T_x^f(x) \);
(b) for each \( x \in X \), \( V(x) \in T_x^h(x) \);
(c) for each \( x \in X \), \( \lim_{t \to 0^+} t^{-1}d_x(x + tV(x)) = 0 \);
(d) for each \( x \in X \), \( \liminf_{t \to 0^+} t^{-1}d_x(x + tV(x)) = 0 \).

**Proof.** The implications (a) \( \Rightarrow \) (b), (a) \( \Rightarrow \) (c), (c) \( \Rightarrow \) (d), (d) \( \Leftrightarrow \) (b) are obvious. Suppose (b) is satisfied. Then for each \( x \in X \) we have \( V(x) = \lim_{y \to x, \, y \in X} V(y) \), hence \( V(x) \in \liminf_{y \to x, \, y \in X} T_x^h(y) \subset T_x^h(x) \), hence (a) holds true. \( \square \)

**Added in proof.** Since the present paper has been submitted for publication, the inclusion of Corollary 2 has been proved by different methods and in an independent way in references [18 and 19] below. Both references contain counterexamples showing that the inclusion may be strict.

**References**


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Département de Mathématiques, Université de Pau, Avenue de L’Université 64000, Pau, France