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MEAN EXIT TIME OF A DIFFUSION PROCESS FROM A SMALL SPHERE

MARK A. PINSKY[1]

ABSTRACT. The first two nonzero terms in the asymptotic expansion of the mean exit time from a small sphere are expressed in terms of the radius, the scalar curvature of the Riemannian metric and the vector field which defines the drift in the infinitesimal generator.

Consider a nondegenerate diffusion process in R^n. Let T_ε be the exit time from a ball of radius ε, where we use the Riemannian distance determined by the second order coefficients. In this note we compute the first two nonzero terms in the asymptotic expansion of $E_m(T_\varepsilon)$, $\varepsilon \downarrow 0$. We have the following result:

THEOREM. *When* $\varepsilon \downarrow 0$

$$E_m(T_\varepsilon) = \frac{\varepsilon^2}{2n} + \frac{\varepsilon^4}{12n^2(n+2)}\left(\tau_m - 3(\operatorname{div} b)_m - \frac{3}{2}|b|_m^2\right) + O(\varepsilon^6),$$

where τ_m *is the scalar curvature of the Riemannian metric* (g_{ij}) *and* (b_i) *is the vector field which appears when the infinitesimal generator is written as* $L = \Delta + b \cdot \nabla$, *or in coordinate form*

$$L = \sum_{ij} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x_j}\right) + \sum_i b_i \frac{\partial}{\partial x_i}$$

with $g^{ij} = (g^{-1})_{ij}$ *and* $g = \det(g_{ij})$.

PROOF OF THE THEOREM. We use the perturbation method initiated in [1]. In a system of normal coordinates $(x_1, \ldots, x_n)$ at $m \in \mathbf{R}^n$ we have

$$\Delta = \Delta_{-2} + \Delta_0 + \cdots, \qquad b \cdot \nabla = B_{-1} + B_0 + \cdots,$$

where

$$\Delta_{-2} = \sum_i \frac{\partial^2}{\partial x_i^2},$$

$$\Delta_0 = \frac{1}{3} \sum_{iajb} R_{iajb} x_a x_b \frac{\partial^2}{\partial x_i \partial x_j} - \frac{2}{3} \sum_{ia} \rho_{ia} x_a \frac{\partial}{\partial x_i},$$

$$B_{-1} = \sum_i b_i \frac{\partial}{\partial x_i}, \qquad B_0 = \sum_{ij} b_{ij} x_j \frac{\partial}{\partial x_i}$$

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and (R_{iajb}) is the curvature tensor at $m \in \mathbf{R}^n$, $\rho_{ia} = \sum_j R_{jija}$ is the Ricci tensor and $b_i = b_i(m)$, $b_{ij} = \partial b_i(m)/\partial x_j$. The mean exit time is the solution of $Lf = -1$ which vanishes on the ε-sphere. We obtain an approximate solution in the form $\bar{f}_\varepsilon(x) = \varepsilon^2 f_0(x/\varepsilon) + \varepsilon^3 f_1(x/\varepsilon) + \varepsilon^4 f_2(x/\varepsilon)$, where

$$\begin{aligned}
&\Delta_{-2} f_0 + 1 = 0,\\
&\Delta_{-2} f_1 + B_{-1} f_0 = 0,\\
&\Delta_{-2} f_2 + B_{-1} f_1 + (\Delta_0 + B_0) f_0 = 0
\end{aligned}$$

and the functions f_0, f_1, f_2 are required to be zero on the unit sphere. Solving explicitly, we have

$$f_0 = \frac{1 - |x|^2}{2n}, \qquad f_1 = \frac{1}{2n(n+2)}(1 - |x|^2) \sum_i b_i x_i,$$

$$B_{-1} f_1 + B_0 f_0 = \frac{(\Sigma b_i x_i)^2}{n(n+2)} - \frac{(1 - |x|^2)}{2n(n+2)} |b|^2 - \frac{1}{n}\left(\sum_{ij} b_{ij} x_i x_j\right),$$

$$\Delta_0 f_0 = -\frac{1}{3n} \sum_{ij} \rho_{ij} x_i x_j.$$

Applying Lemma 6.3 of **[1]**, we have

$$\begin{aligned}
f_2(0) &= \frac{\tau_m}{12n^2(n+2)} + \frac{1}{n(n+2)} \frac{|b|_m^2}{4(n+2)n}\\
&\quad - \frac{|b|_m^2}{2n(n+2)}\left[\frac{1}{2n} - \frac{1}{4(n+2)}\right] - \frac{(\operatorname{div} b)_m}{4n^2(n+2)}\\
&= \frac{1}{12n^2(n+2)}\left(\tau_m - 3(\operatorname{div} b)_m - \frac{3}{2}|b|_m^2\right).
\end{aligned}$$

The proof is complete.

References

1. A. Gray and M. Pinsky, *The mean exit time from a small geodesic ball in a Riemannian manifold*, Bull. Sci. Math. (2) **107** (1983), 345–370.

Department of Mathematics, Northwestern University, Evanston, Illinois 60201