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INSERTION, APPROXIMATION, AND EXTENSION OF REAL-VALUED FUNCTIONS

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ABSTRACT. For a uniformly closed vector lattice V of real-valued functions on a set X, necessary and sufficient conditions are obtained for insertion (or "strict insertion") of some member of V between two arbitrary real-valued functions on X. These conditions quickly yield known insertion, approximation, and extension theorems for real-valued functions.

1. Introduction. For X a set, $F(X)$ (resp. $F^*(X)$) denotes the set of all real-valued (resp. bounded real-valued) functions on X. We view $F(X)$ as both a vector lattice and ring (with the pointwise operations) and as a topological space with the topology of uniform convergence. A "vector sublattice" or "subring" of $F(X)$ will always be understood to contain all the constant real-valued functions on X.

For $f \in F(X)$ and $a \in \mathbf{R}$, set $L_a(f) = \{x \in X\colon f(x) \leq a\}$, $L^a(f) = \{x \in X\colon f(x) \geq a\}$, and $Z(f) = \{x \in X\colon f(x) = 0\}$. Sets of the form $L_a(f)$ (resp. $L^a(f)$) are *lower* (resp. *upper*) *Lebesgue sets* of f.

For V a vector sublattice of $F(X)$, $\operatorname{uc} V$ denotes the uniform closure of V in $F(X)$ and (following [**Ma**, p. 51]) $\operatorname{us} V$ (resp. $\operatorname{ls} V$) denotes the set of all $f \in F(X)$ such that f is the pointwise limit of some decreasing (resp. increasing) sequence of functions in V. We say that V is *inversion-closed* (resp. *closed under* (*bounded*) *quotients*) if $1/g \in V$ whenever $g \in V$ with $Z(g) = \varnothing$ (resp. $f/g \in V$ whenever $f, g \in V$ (with f/g bounded) and $Z(g) = \varnothing$). If $f, g \in F(X)$, then V *completely separates the Lebesgue sets of the pair* $\langle f, g\rangle$ if for every $a < b$ in $\mathbf{R}$, $L_a(g)$ and $L^b(f)$ are completely separated by some function in V (i.e. there exists $k \in V$ such that $k = 0$ on one of the sets $L_a(g)$ and $L^b(f)$, $k = 1$ on the other, and $0 \leq k \leq 1$), and V *completely separates the Lebesgue sets of* f if V completely separates the Lebesgue sets of $\langle f, f\rangle$. By a V*-zero-set* we mean a set of the form $Z(f)$ with $f \in V$.

For a topological space X and $S \subset X$, set $C(X) = \{f \in F(X)\colon f \text{ is continuous}\}$, $C^*(X) = C(X) \cap F^*(X)$, $C(X)|S = \{f|S\colon f \in C(X)\}$, and $C^*(X)|S = \{f|S\colon f \in C^*(X)\}$. As usual, S is *C-embedded* (resp. *C^*-embedded*) in X if $C(S) = C(X)|S$ (resp. $C^*(S) = C^*(X)|S$). By a *zero-set* of X we mean a $C(X)$-zero-set.

We begin by quoting two basic results:

1.1. APPROXIMATION THEOREM. *Let X be a set, let V be a vector sublattice of $F(X)$, and let $f \in F(X)$. If V completely separates the Lebesgue sets of f, and if either $f \in F^*(X)$ or $\operatorname{uc} V$ is inversion-closed, then $f \in \operatorname{uc} V$.*

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1.2. TOPOLOGICAL INSERTION THEOREM. *If X is a topological space and if $f, g \in F(X)$, then the following are equivalent:*

(1) *There exists $h \in C(X)$ such that $f \leq h \leq g$.*

(2) *For every $a < b$ in $\mathbf{R}$, $L_a(g)$ and $L^b(f)$ are completely separated in X.*

1.3. REMARKS. (a) 1.1 is due to Mrówka (for $f \in F^*(X)$ see [**M**, 2.7] and for $\mathrm{uc}\, V$ inversion-closed see [**M**, 3.3] and 2.2 below). Consequences of 1.1 include the Extension Theorem (1.4 below) as well as several versions of the Stone-Weierstrass Theorem (see e.g. [**M**, 4.1, 4.6, and 4.8]).

(b) For an example of an inversion-closed vector lattice of functions whose uniform closure is not inversion-closed, see 3.6 below.

(c) 1.2 is due independently to Blair [**Bl**, 3.5] and Lane [$\mathbf{L}_1$, 2.1]. (Its proofs in [**Bl**] and [$\mathbf{L}_1$] are based on techniques of Tong [**T**] and Katětov [$\mathbf{K}_1$] (correction [$\mathbf{K}_2$]), respectively.) 1.2 generalizes earlier results of Katětov [$\mathbf{K}_1$, $\mathbf{K}_2$] and Tong [**T**, Theorem 2] for normal spaces (cf. Engelking [**E**, 1.7.15(b)] and Bourbaki [**Bo**, Chapter IX, §4, Exercise 30]), Dieudonné [**D**, Theorem 9] for paracompact spaces, and Hahn [**H**, p. 103] for metric spaces.

We next record a basic extension theorem. It is due to Mrówka [**M**, 4.11] and obviously implies the Gillman-Jerison version of Urysohn's Extension Theorem [**GJ**, 1.17] (namely, that a subset S of a space X is C^*-embedded in X if and only if any two completely separated subsets of S are completely separated in X).

1.4. EXTENSION THEOREM. *If X is a topological space, $S \subset X$, and $f \in C^*(S)$, then the following are equivalent:*

(1) *f has a continuous extension over X.*

(2) *For every $a < b$ in $\mathbf{R}$, $L_a(f)$ and $L^b(f)$ are completely separated in X.*

We note that the nontrivial implication (2)$\Rightarrow$(1) of 1.4 is a consequence of either 1.1 or 1.2. Its easy proof from 1.2 is given in [**Bl**, p. 67], and its proof from 1.1 (as in [**M**]) is immediate in view of 1.5(1):

1.5. PROPOSITION. *Let X be a topological space and let $S \subset X$.*

(1) *$C(X)|S$ is uniformly closed in $F(S)$* [**M**, 4.10].

(2) *$C(X)|S$ is inversion-closed if and only if S is completely separated from every zero-set of X disjoint from S* [**M**, 4.15].

For a self-contained proof of 1.4, see [**Bl**, 3.2]. We note also that, in view of 1.5, 1.1 quickly yields the nontrivial half of the Gillman-Jerison characterization of C-embedding [**GJ**, 1.18] (namely, that a subset S of a space X is C-embedded in X if and only if S is C^*-embedded in X and completely separated from every zero-set of X disjoint from S).

To summarize, we have the following diagram:

$$\begin{tikzcd} \text{Approximation Theorem} \arrow[dr, Rightarrow] & & \text{Topological Insertion Theorem} \arrow[dl, Rightarrow] \\ & \text{Extension Theorem} & \end{tikzcd}$$

The principal object of this paper is to complete the foregoing diagram with the following topology-free insertion theorem, which quickly yields both the Approximation Theorem and the Topological Insertion Theorem.

1.6. INSERTION THEOREM. *Let X be a set, let V be a uniformly closed vector sublattice of $F(X)$, and let $f, g \in F(X)$. If either $f, g \in F^*(X)$ or V is inversion-closed, then the following are equivalent:*

(1) *There exists $h \in V$ such that $f \leq h \leq g$.*

(2) *V completely separates the Lebesgue sets of $\langle f, g\rangle$.*

We prove 1.6 in §2. Note that if V is a vector sublattice of $F(X)$, then so is uc$V$, and hence 1.1 follows immediately from 1.6. Moreover, since $C(X)$ is a uniformly closed inversion-closed vector sublattice of $F(X)$, 1.2 is also an immediate consequence of 1.6.

Some results on "strict insertion" (which are related to 1.6 and which generalize results of Lane [$\mathbf{L}_2$]) are obtained in §3.

2. Proof of the Insertion Theorem. We first prove three lemmas.

2.1. LEMMA. *If V is a vector sublattice of $F(X)$, if $f, g \in F^*(X)$, and if V completely separates the Lebesgue sets of $\langle f, g\rangle$, then there exists a decreasing sequence $\langle f_n : n \in \omega\rangle$ of functions in V such that:*

(1) *For every $n \in \omega$, $f_n - g \leq 2^{-n}$.*

(2) *$f \leq \bigwedge_{n\in\omega} f_n \leq g$.*

(3) *V completely separates the Lebesgue sets of $\langle \bigwedge_{n\in\omega} f_n, g\rangle$.*

PROOF. It is easy to see that $f \leq g$, and thus there exist integers p and q such that $p \leq f \leq g \leq q$. For each $n \in \omega$ and each $m \in \omega$ with $m < (q-p)2^{n+1}$, the hypotheses on V imply that there exists $f_{mn} \in V$ such that

$$f_{mn} = p + (m+1)2^{-n-1} \quad \text{on } \{x \in X\colon\ g(x) \leq p + m2^{-n-1}\},$$
$$f_{mn} = q \quad \text{on } \{x \in X\colon\ f(x) \geq p + (m+1)2^{-n-1}\},$$

and

$$p + (m+1)2^{-n-1} \leq f_{mn} \leq q.$$

For each $n \in \omega$, let $f_n^* = \bigwedge\{f_{mn}\colon\ m < (q-p)2^{n+1}\}$, let $f_n = \bigwedge_{i\leq n} f_i^*$, and note that $\langle f_n : n \in \omega\rangle$ is a decreasing sequence of functions in V such that $f \leq f_n$ for every $n \in \omega$. Thus $f \leq \bigwedge_{n\in\omega} f_n$.

To verify (1), let $n \in \omega$ and $x \in X$. Then $f_n^*(x) = f_{mn}(x)$ for some $m < (q-p)2^{n+1}$. Now either $g(x) > q - 2^{-n-1}$ or $p + k2^{-n-1} \leq g(x) \leq p + (k+1)2^{-n-1}$ for some $k \in \omega$ with $k+1 < (q-p)2^{n+1}$. In the first case,

$$f_n(x) - g(x) \leq f_{mn}(x) - g(x) < q - q + 2^{-n-1} < 2^{-n},$$

and in the second,

$$f_n(x) - g(x) \leq f_{k+1,n}(x) - g(x) \leq p + (k+2)2^{-n-1} - (p + k2^{-n-1}) = 2^{-n}.$$

Thus (1) holds, and from (1) we conclude that $\bigwedge_{n\in\omega} f_n \leq g$.

Finally, to verify (3), let $a, b \in \mathbf{R}$ with $a < b$ (and we may assume that $p \leq a < b \leq q$). Pick $k \in \omega$ with $2^{-k} < b - a$ and let

$$m = \max\{n \in \omega\colon\ p + n2^{-k-1} \leq (a+b)/2\}.$$

Then $a < p + m2^{-k-1} < p + (m+1)2^{-k-1} < b$, $L_a(g) \subset \{x \in X\colon\ f_{mk}(x) = p + (m+1)2^{-k-1}\} \subset \{x \in X\colon\ f_k(x) \leq p + (m+1)2^{-k-1}\}$, and $L^b(\bigwedge_{n\in\omega} f_n) \subset L^b(f_k)$. But since $f_k \in V$, it is clear that V completely separates the Lebesgue sets of f_k, and thus $L_a(g)$ and $L^b(\bigwedge_{n\in\omega} f_n)$ are completely separated by some function in V. $\square$

For the proof of the following lemma, see e.g. [**Ha**, 2.4 and **M**, Remark 2 and 3.2].

2.2. LEMMA. *If V is a uniformly closed subset of $F(X)$ and if either $V \subset F^*(X)$ or V is inversion-closed, then V is a vector lattice if and only if V is a ring.*

We define the map γ: $\mathbf{R} \to (-1,1)$ by the formula $\gamma(t) = t/(1+|t|)$ for $t \in \mathbf{R}$. Clearly γ is an order-preserving homeomorphism from $\mathbf{R}$ onto the open interval $(-1,1)$, and its inverse is given by the formula $\gamma^{-1}(s) = s/(1-|s|)$ for $s \in (-1,1)$. By 2.2 we have

2.3. LEMMA. *Let V be a uniformly closed inversion-closed vector sublattice of $F(X)$. Then V is closed under quotients, and if $h \in V$ with $h(X) \subset (-1,1)$, then $\gamma^{-1} \circ h \in V$.*

PROOF OF 1.6. (1)$\Rightarrow$(2). Let h be as in (1) of 1.6. Since V completely separates the Lebesgue sets of h, (2) follows from the inequalities $f \le h \le g$.

(2) $\Rightarrow$ (1). We first consider the case in which $f, g \in F^*(X)$. By (2) and 2.1, there exists a decreasing sequence $\langle f_n \colon n \in \omega\rangle$ of functions in V that satisfies (2) and (3) of 2.1. Then clearly V completely separates the Lebesgue sets of $\langle -g, -\bigwedge_{n\in\omega} f_n\rangle$, so by 2.1 again there is a decreasing sequence $\langle g'_n \colon n \in \omega\rangle$ of functions in V such that $g'_m + \bigwedge_{n\in\omega} f_n \le 2^{-m}$ for every $m \in \omega$ and $-g \le \bigwedge_{n\in\omega} g'_n \le -\bigwedge_{n\in\omega} f_n$. Set $g_n = -g'_n$. Then $\langle g_n \colon n \in \omega\rangle$ is an increasing sequence of functions in V such that $f \le \bigwedge_{n\in\omega} f_n \le \bigvee_{n\in\omega} g_n \le g$ and

$$(*) \qquad \bigwedge_{n\in\omega} f_n \le g_m + 2^{-m} \quad \text{for every } m \in \omega.$$

Next, there exists (recursively) a sequence $\langle h_n \colon n \in \omega\rangle$ of functions in V such that $h_0 = f_0 \wedge g_0$ and such that, for every $n > 0$,

$$h_n = \left(\bigvee_{i\le n} (f_i \wedge g_i)\right) \wedge (h_{n-1} + 2^{-n}).$$

Note that for every $n \in \omega$, $h_n \le \bigvee_{i\le n} g_i \le g$ and that (inductively) $h_n \le h_{n+1}$. Set $h = \bigvee_{n\in\omega} h_n$ and observe that $h \le g$.

We claim that $h \in V$. For this it suffices to show that the (increasing) sequence $\langle h_n \colon n \in \omega\rangle$ converges uniformly to h; and hence it suffices to show that $\langle h_n \colon n \in \omega\rangle$ is uniformly Cauchy: For $\varepsilon > 0$, pick $m \in \omega$ with $\sum_{i=m+1}^{\infty} 2^{-i} < \varepsilon$ and note that if $m \le j < k$, then

$$|h_k - h_j| \le \sum_{i=j}^{k-1} |h_{i+1} - h_i| \le \sum_{i=j}^{k-1} 2^{-i-1} < \varepsilon.$$

It remains only to show that $f \le h$. Let $x \in X$. If $h_k(x) \ge f_k(x)$ for some $k \in \omega$, then $f(x) \le f_k(x) \le h(x)$, so we may assume that $h_k(x) < f_k(x)$ for every $k \in \omega$. Moreover, if for every $n \in \omega$ there exists $k \ge n$ such that $h_k(x) = g_k(x)$, then $f(x) \le \bigvee_{n\in\omega} g_n(x) = \bigvee_{n\in\omega} h_n(x) = h(x)$, so we may also assume that there exists a (least) $m \in \omega$ such that $h_k(x) \ne g_k(x)$ for every $k \ge m$. Clearly $m > 0$

and, by induction, $h_n(x) = g_{m-1}(x) + \sum_{i=m}^{n} 2^{-i}$ for every $n \geq m$. Then by $(*)$ we have

$$f(x) \leq \bigwedge_{n\in\omega} f_n(x) \leq g_{m-1}(x) + 2^{-m+1}$$
$$= g_{m-1}(x) + \sum_{i=m}^{\infty} 2^{-i} = \bigvee_{n\geq m} h_n(x) = h(x),$$

and we conclude that $f \leq h$.

Now assume that V is inversion-closed and that (merely) $f, g \in F(X)$. Since $\gamma \circ f$, $\gamma \circ g \in F^*(X)$ and V completely separates the Lebesgue sets of $\langle \gamma \circ f, \gamma \circ g \rangle$, by what was just proved there exists $h \in V$ such that $\gamma \circ f \leq h \leq \gamma \circ g$. Then $\gamma^{-1} \circ h \in V$ by 2.3, and $f \leq \gamma^{-1} \circ h \leq g$. □

2.4. REMARKS. Parts of the preceding proof were suggested by the Tong-Blair proof of 1.2 (see 1.3(c)). But Tong's technique (in the proof of [**T**, Theorem 2]) is ultimately inadequate for a proof of (2)⇒(1) of 1.6 since it merely produces an $h \in \operatorname{us} V \cap \operatorname{ls} V$ with $f \leq h \leq g$, and such an h need not be in V, as the following example shows: Let $X = (0,1]$, define $h \in F(X)$ by $h(x) = \sin(1/x)$, and let $V = C(\mathbf{R})|X$. Then V is a uniformly closed vector sublattice of $F(X)$ (by 1.5(1)), $h \in \operatorname{us} V \cap \operatorname{ls} V$ (as is easily seen), but $h \notin V$. (The Katětov-Lane proof of 1.2 (see 1.3(c)) also does not appear to be readily adaptable to a proof of 1.6.)

3. Strict insertion. For $f, g, h \in F(X)$, h is *strictly between f and g* if $f \leq h \leq g$ and $f(x) < h(x) < g(x)$ whenever $x \in X$ with $f(x) < g(x)$.

Our first result on "strict insertion" is an easy corollary of 1.6:

3.1. THEOREM. *Let V be a uniformly closed vector sublattice of $F(X)$ and let $f, g \in F(X)$. If either $f, g \in F^*(X)$ or V is inversion-closed, then the following are equivalent:*

(1) *There exists $h \in V$ such that h is strictly between f and g.*

(2) *There exists $k \in F(X)$ such that k is strictly between f and g and such that V completely separates the Lebesgue sets of both $\langle f, k \rangle$ and $\langle k, g \rangle$.*

PROOF. (1) ⇒ (2). It suffices to take $k = h$.

(2) ⇒ (1). By (2) and 1.6 there exist $h_1, h_2 \in V$ such that $f \leq h_1 \leq k \leq h_2 \leq g$. Then $h = (h_1 + h_2)/2$ is in V and h is strictly between f and g. □

In what follows we generalize techniques and results of Lane [**L**$_2$].

The next lemma is for the most part implicit in the proof of Lane [**L**$_2$, Theorem 2], and in the bounded case is also a consequence of Mauldin [**Ma**, Theorem 4] (cf. Hager [**Ha**, p. 762]). It generalizes earlier results of Tong for perfectly normal spaces [**T**, Theorem 3], Hahn for metric spaces [**H**, p. 100], and Baire for the real line [**Ba**, p. 125]. We give a brief proof based on the proof of [**T**, Theorem 3].

3.2. LEMMA. *Let V be a uniformly closed vector sublattice of $F(X)$ and let $f \in F(X)$. If either $f \in F^*(X)$ or V is inversion-closed, then the following are equivalent:*

(1) *Every upper (resp. lower) Lebesgue set of f is a V-zero-set.*

(2) $f \in \operatorname{us} V$ *(resp.* $f \in \operatorname{ls} V$*).*

PROOF. (1) ⇒ (2). If $f \in F^*(X)$, then there exists $r \in \mathbf{R}$ with $r > 0$ and $f(X) \subset (-r, r)$. Define (the order-preserving homeomorphism) τ: $(-r, r) \to (0, 1)$

by the formula $\tau(t) = (r+t)/2r$ for $t \in (-r, r)$. By (1) and the proof of (1)$\Rightarrow$(2) of [**T**, Theorem 3], there exists a decreasing (resp. increasing) sequence $\langle f_n\colon n \in \omega\rangle$ of functions in V whose pointwise limit is $\tau \circ f$ and with $f_n(X) \subset (0,1)$ for every $n \in \omega$. Then $\langle \tau^{-1} \circ f_n\colon n \in \omega\rangle$ is a decreasing (resp. increasing) sequence of functions in V whose pointwise limit is f.

If, on the other hand, V is inversion-closed, then by 2.3 and what was just proved (with $r = 1$), $\langle \gamma^{-1} \circ \tau^{-1} \circ f_n\colon n \in \omega\rangle$ is again a decreasing (resp. increasing) sequence of functions in V whose pointwise limit is f.

(2) $\Rightarrow$ (1). Assume there is a decreasing sequence $\langle f_n\colon n \in \omega\rangle$ of functions in V with pointwise limit f. Let $a \in \mathbf{R}$, for each $n \in \omega$ set $g_n = 1 \wedge (a - (f_n \wedge a))$, and let $g = \sum_{n\in\omega} 2^{-n} g_n$. Then $g \in V$ and $L^a(f) = \bigcap_{n\in\omega} Z(g_n) = Z(g)$. The other case is proved similarly. $\square$

3.3. REMARKS. (a) The equivalence (ii)$\Leftrightarrow$(iii) of [$\mathbf{L}_2$, Theorem 2] is essentially the special case of 3.2 for which X is a topological space, $f \in F^*(X)$, and $V = C^*(X)$.

(b) For $f \in F^*(X)$, (1) $\Rightarrow$ (2) of 3.2 holds even if V is not uniformly closed (as can be seen from the proof of [**T**, Theorem 3]); and in fact (1) $\Rightarrow$ (2) holds if f is merely bounded above (resp. below) (see [**Ma**, Theorem 4]).

The following lemma is clear.

3.4. LEMMA. *If V is a vector sublattice of $F(X)$ that is closed under bounded quotients and if $u, v \in V$ with $Z(u) \cap Z(v) = \varnothing$, then the function $|u|/(|u| + |v|)$ is in V and completely separates $Z(u)$ and $Z(v)$.*

3.5. COROLLARY. *If V is a vector sublattice of $F(X)$ and if V is closed under bounded quotients (resp. if $\operatorname{uc} V$ is inversion-closed), then $\operatorname{us} V \cap \operatorname{ls} V \cap F^*(X) \subset \operatorname{uc} V$ (resp. $\operatorname{us} V \cap \operatorname{ls} V \subset \operatorname{uc} V$).*

PROOF. This is immediate from 3.2, 3.4, 1.1, and 2.3. $\square$

3.6. EXAMPLE. We give an example of a set X and an inversion-closed vector sublattice V of $F(X)$ such that $\operatorname{us} V \cap \operatorname{ls} V \not\subset \operatorname{uc} V$ (and hence with $\operatorname{uc} V$ not inversion-closed by 3.5): Let X and h be as in 2.4. Clearly there exist increasing and decreasing sequences $\langle f_n\colon n \in \omega\rangle$ and $\langle g_n\colon n \in \omega\rangle$ of functions in $C(X)$, both pointwise convergent to h, such that each f_n (resp. g_n) is -1 (resp. 1) on some neighborhood of 0. Let W be the smallest lattice-ordered subring of $F(X)$ that contains $\{f_n\colon n \in \omega\} \cup \{g_n\colon n \in \omega\}$ and let $V = \{p/q\colon p, q \in W \text{ and } Z(q) = \varnothing\}$. One can verify that V is an inversion-closed vector sublattice of $F(X)$ and that each function in V is constant on some neighborhood of 0. Hence $V \subset C(\mathbf{R})|X$, and therefore $\operatorname{uc} V \subset C(\mathbf{R})|X$ by 1.5(1). Thus $h \in (\operatorname{us} V \cap \operatorname{ls} V) - \operatorname{uc} V$.

Our main result on strict insertion is as follows:

3.7. THEOREM. *Let V be a uniformly closed vector sublattice of $F(X)$ and let $f, g \in F(X)$. If either $f, g \in F^*(X)$ and V is closed under bounded quotients, or if V is inversion-closed, then the following are equivalent:*

(1) *There exists $h \in V$ such that h is strictly between f and g.*

(2) *There exist $f', g' \in F(X)$ such that every upper (resp. lower) Lebesgue set of f' (resp. g') is a V-zero-set, $f \leq f' \leq g' \leq g$, and $f'(x) < g(x)$ and $f(x) < g'(x)$ whenever $x \in X$ with $f(x) < g(x)$.*

PROOF. We assume first that $f, g \in F^*(X)$ and that V is closed under bounded quotients.

(1) $\Rightarrow$ (2). It suffices to take $f' = h = g'$.

(2) $\Rightarrow$ (1). By 3.4 and the hypotheses on the Lebesgue sets of f' and g', V completely separates the Lebesgue sets of $\langle f', g'\rangle$ and hence also those of $\langle f, g\rangle$. By 1.6, there exists $k \in V$ such that $f \leq k \leq g$. Moreover, by 3.2 there exist, respectively, decreasing and increasing sequences $\langle f_n \colon n \in \omega\rangle$ and $\langle g_n \colon n \in \omega\rangle$ of functions (which we may assume are bounded) in V such that $f' = \bigwedge_{n\in\omega} f_n$ and $g' = \bigvee_{n\in\omega} g_n$. Then (1) follows from (2) by the argument of the proof of [$\mathbf{L}_2$, Lemma 1].

The case in which V is inversion-closed now follows from what has just been proved together with 2.3. $\square$

3.8. COROLLARY. *Let V be a uniformly closed vector sublattice of $F(X)$, let $f, g \in F(X)$ with $f \leq g$, and assume that either $f, g \in F^*(X)$ and V is closed under bounded quotients or that V is inversion-closed. If every upper (resp. lower) Lebesgue set of f (resp. g) is a V-zero-set, then there exists $h \in V$ such that h is strictly between f and g.*

3.9. REMARKS. (a) The implication (ii)$\Rightarrow$(i) of [$\mathbf{L}_2$, Theorem 2] is essentially the special case of 3.8 for which X is a topological space, $f, g \in F^*(X)$, and $V = C^*(X)$.

(b) Mauldin proves the following in [**Ma**, Theorem 6]: $(*)$ If V is a vector sublattice of $F(X)$ and if $f \in \operatorname{us} V$ and $g \in \operatorname{ls} V$ with $f \leq g$, then there exists $h \in \operatorname{us} V \cap \operatorname{ls} V$ with h strictly between f and g. We note that $(*)$ immediately yields the special case of 3.8 described in (a) and that 3.8 itself follows from $(*)$, 3.2, and 3.5.

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