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A NECESSARY AND SUFFICIENT CONDITION FOR A CONNECTED AMENABLE GROUP TO HAVE POLYNOMIAL GROWTH

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ABSTRACT. It is shown that a connected amenable group G has polynomial growth if, and only if, given any open subsemigroup S of G and a compact set K in G there exists an s in S such that $Ks \subset S$.

Introduction. Let G be a connected topological group and $C(G)$ the space of all bounded continuous real-valued functions on G. For each f in $C(G)$ and a in G define $\|f\| = \sup\{|f(x)|: x \in G\}$ and $l_a f(t) = f(a^{-1}t)$ for all t in G. A function f in $C(G)$ is left uniformly continuous if whenever (s_i) is a net in G and converges to some s in G, then $\|l_{s_i} f - l_s f\|$ tends to zero. We say that G is amenable (more accurately, left amenable) if there exists an m in $\mathrm{LUC}(G)^*$, the conjugate space of $\mathrm{LUC}(G)$ (the space of left uniformly continuous functions), such that $m \geq 0$, $\|m\| = 1$ and $m(l_a f) = m(f)$ for all f in $\mathrm{LUC}(G)$ and a in G. The reader is referred to Greenleaf [**3**] for many interesting properties of amenable groups. Another interesting class of groups are those with polynomial growth, whose definition follows. Let $|A|_G$ denote the left Haar measure of a measurable subset A of G. A locally compact group G is said to have polynomial growth if for each compact nbd U of the identity e in G, there exists a polynomial p such that $|U^n|_G \leq p(n)$ for all $n = 1, 2, 3, \ldots$.

The notion of Archimedean property for groups was introduced in [**2**]. A group G is said to have Archimedean property if whenever S is an open generating subsemigroup and K a compact subset of G there exists an s in S such that $sK \subset S$. It is proved in [**2**] that any connected group with polynomial growth has Archimedean property. Some remarkably diverse properties of groups with polynomial growth can be found in [**1, 4, 7**]. It is well known that groups with polynomial growth are amenable. However, the converse fails. The group of affine transformations on the real line is a connected solvable group without polynomial growth. Jenkins [**6**] has given a necessary and sufficient condition for a connected amenable group to have polynomial growth. In this paper we give another necessary and sufficient condition in Theorem 2. As a corollary we note that every bounded continuous function on a group with polynomial growth which is left uniformly continuous on a dense open subsemigroup is left uniformly continuous on G.

First we prove the following

PROPOSITION 1. *Let G be a connected locally compact group and S an open subsemigroup of G. Then the following are equivalent.*

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(a) *The open right ideals of S have finite intersection property, i.e., any finite set of nonempty open right ideals of S has nonempty intersection.*

(b) $G = SS^{-1}$.

(c) *Given any compact set K in G there exists an s in S such that $Ks \subseteq S$.*

PROOF. (b) implies (c). First, we prove that given any finite subset F of G there exists a t in S such that $Ft \subseteq S$. If $F = \{k_1, k_2, \ldots, k_n\}$ there exists an s_1 in S such that $k_1 s_1$ is in S; there exists an s_2 in S with $(k_2 s_1)$ in S, etc. Then $Fs_1 s_2 \cdots s_n$ is a subset of S. Now, if K is any compact subset (given) fix an s in S. Let $U = U^{-1}$ be a nbd of e with compact closure such that $A = sU$ is a subset of S. Consider the compact set $s^{-1}K$; since G is connected, there is a large N such that $s^{-1}K \subset U^N$. Since U is an open set with compact closure there exists a finite set F such that $UU \subset \overline{UU} \subset UF$ and, choosing t as above, $U^N t \subset UF^{N-1}t \subset US$. Hence, $Kt \subset sU^N t \subset sUS \subset SS \subset S$, which proves (c).

The proof of (c) implies (b) is obvious.

(b) implies (a). Suppose I and J are two open right ideals of S which are disjoint. Consider $b^{-1}a$ in G where $a \in I$, $b \in J$; then $b^{-1}a = s_1 s_2^{-1}$ for some s_1 and s_2 in S. Hence, $(bs_1)^{-1} = s_1^{-1}b^{-1} = s_2^{-1}a^{-1} = (as_2)^{-1}$, so that $I^{-1} \cap J^{-1} \neq \emptyset$, a contradiction.

(a) implies (b). SS^{-1} is an open subgroup of G containing e, the identity of G. For, if a, b, c and d are in S then $ab^{-1}cd^{-1} \in SS^{-1}$. Since cS and bS are open right ideals in S, cS and bS have an element in common, i.e., there exists s_1, s_2 in S such that $cs_1 = bs_2$, which implies $ab^{-1}cd^{-1} = as_2(ds_1)^{-1} \in SS^{-1}$. This shows that SS^{-1} is an open subgroup and hence closed. Since G is connected, $G = SS^{-1}$.

With very few modifications to [**2**, Lemma 7] one can show that any connected locally compact group with polynomial growth has property (c) of the above proposition. Also, note that $G = S^{-1}S$ for any open subsemigroup S of G is equivalent to: given any compact set K in G there exists an s in S such that $sK \subset S$. The above two remarks imply that if G has polynomial growth then for any open subsemigroup S of G, $G = SS^{-1} = S^{-1}S$.

Jenkins [**5, 6**] has shown that a connected amenable group G has polynomial growth if, and only if, each open subsemigroup of G is amenable if, and only if, the open right ideals of an open subsemigroup of G have finite intersection property. Using the above remark and the Proposition we have

THEOREM 2. *Let G be a connected group. Then the following are equivalent.*

(a) *G is an amenable group with the property that given any compact set K in G and any open semigroup S of G, then there exists an s in S such that $Ks \subset S$.*

(b) *G has polynomial growth.*

The following is worth noting.

COROLLARY. *Let G be a connected group with polynomial growth and S an open dense subsemigroup of G. If F is a bounded continuous function on G such that $F_{|S}$ is in* LUC(S), *then F is in* LUC(G).

PROOF. Since G has polynomial growth, it is an Archimedean group and $G = SS^{-1}$. The rest of the proof follows from Lau [**8**].

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