A GENERALIZED JACOBIAN CRITERION FOR REGULARITY

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ABSTRACT. For a commutative noetherian algebra B over a perfect field A regularity is equivalent to the flatness of $\Omega_{B/A}$ plus $H_1(A, B, B) = 0$ (simplicial homology). In characteristic 0 the homological condition is superfluous.

1. Introduction. All rings to be considered will be commutative and unitary. The classical Jacobian criterion (in terms of Kähler differentials) reads like this: Let A be a perfect field, $(B, m, K)$ a local A-algebra of finitely generated type (obtained by localizing an A-algebra of finite type) such that $K = B/m = A$ and let $\Omega_{B/A}$ be the B-module of A-differentials for B. Then B is regular if and only if $\Omega_{B/A}$ is free (of rank equal to the dimension of B).

I want to generalize this differential characterization of regularity in the following way.

THEOREM. Let A be a perfect field, B a noetherian A-algebra. Then B is regular (all localizations $B_m$ of B at maximal ideals m are regular local rings) if and only if the following two conditions hold:

(1) $\Omega_{B/A}$ is B-flat.

(2) For every infinitesimal A-extension

$$E = 0 \rightarrow M \rightarrow C \rightarrow B \rightarrow 0$$

of B, the associated derived module sequence

$$\text{diff}(E) = 0 \rightarrow M \rightarrow \Omega_{C/A} \otimes_C B \rightarrow \Omega_{B/A} \rightarrow 0$$

is exact.

COROLLARY. If $\text{char } A = 0$, we have

B is regular if and only if $\Omega_{B/A}$ is B-flat.

I do not know whether condition (2) is really relevant in positive characteristic (A perfect, of course).

2. Proof of the theorem. Recall for arbitrary $A \rightarrow B$ the two sequences

$(H_n(A, B, -))_{n \geq 0}$ and $(H^n(A, B, -))_{n \geq 0}$

of simplicial homology and cohomology functors as defined and discussed in [1].
Lemma. For arbitrary $A \to B$ the following three conditions are equivalent:

(a) $\Ext^1_B(\Omega_{B|A}, \cdot) = H^1(A, B, \cdot)$.

(b) $H^1(A, B, B) = 0$.

(c) $H^1(A, B, \cdot) = \Tor^B_1(\Omega_{B|A}, \cdot)$.

Condition (a) needs some comment:

For every $B$-module $M$, $\Ext^1_B(\Omega_{B|A}, M)$ classifies the singular 1-extensions of $\Omega_{B|A}$ by $M$, whereas $H^1(A, B, M)$ classifies the infinitesimal $A$-extensions of $B$ by $M$.

There is a $B$-monomorphism $\Ext^1_B(\Omega_{B|A}, M) \to H^1(A, B, M)$, functorial in $M$, whose image consists of the classes of those $A$-extensions $E = 0 \to M \to C \to B \to 0$, which have an exact derived module sequence

$$\text{diff}(E) = 0 \to M \to \Omega_{C|A} \otimes_C B \to \Omega_{B|A} \to 0$$

(see [3, pp. 158–161]). Thus (a) is only another formulation of condition (2) of the theorem.

Proof of the lemma. (a) implies (b). We have $H^1(A, B, W) = 0$ for every injective $B$-module $W$. But for injective $B$-modules

$$H^1(A, B, W) = \Hom_B(H^1(A, B, B), W)$$

by [1, 3.21, p. 42]. Now take $W$ the injective hull of $H^1(A, B, B)$.

(b) implies (c). This follows from [1, 3.19, p. 41].

(c) implies (a). We have immediately $H^1(A, B, B) = 0$, and thus $H^1(A, B, W) = 0$ whenever $W$ is $B$-injective (see (a) $\Rightarrow$ (b)). But by [2], $H^1(A, B, \cdot)$ is part of an exact connected sequence of cohomology functors, say $(D^n(B|A, \cdot))_{n \geq 0}$, where $D^0(B|A, \cdot) = \Hom_B(\Omega_{B|A}, \cdot)$, $D^1(B|A, \cdot) = H^1(A, B, \cdot)$ and $D^n(B|A, \cdot) = \Ext^1_B(K, \cdot)$, $n \geq 2$, for some $B$-module $K$. The vanishing on injective coefficients in positive degree yields, in particular, $H^1(A, B, \cdot) = \Ext^1_B(\Omega_{B|A}, \cdot)$.

This finishes the proof of the lemma.

Now the theorem follows easily:

By the lemma (condition (b)) we may assume $B$ to be local. Since $A$ is perfect, the regularity of $B$ is equivalent to the formal smoothness of $B$ over $A$. But by [1, Theorem 30, p. 331] this in turn is equivalent to $H^1(A, B, \cdot) = 0$.

The lemma now immediately yields the assertion of the theorem.

As to the corollary, we may rest in the local case. The flatness of $\Omega_{B|A}$ gives the $B|m^k$-projectivity of $\Omega_{B|A} \otimes_B B/m^k$ for every $k \geq 1$ ($m/m^k$ is nilpotent, $m$ the maximal ideal of $B$, of course).

A theorem of Radu's (cf. [1, 7.31, p. 103]) guarantees the regularity of $B$.

References


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