AN ARITHMETIC PROPERTY OF THE TAYLOR COEFFICIENTS OF ANALYTIC FUNCTIONS WITH AN APPLICATION TO TRANSCENDENTAL NUMBERS

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ABSTRACT. We extend a result of Popken concerning the numerators of the Taylor coefficients of algebraic functions and combine it with a result of Mahler on lacunary power series to prove an extension of a special case of a result of Cohn on the transcendence of functional values of lacunary power series evaluated at rational points.

1. Introduction. In 1959 Popken [4] proved the following result [4, Theorems 1, 2, p. 203] for numerators of Taylor coefficients of algebraic functions:

**POPKEN’S THEOREM.** Let the power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), with rational coefficients \( a_n \) \((n = 0, 1, 2, \ldots)\) and convergent in a neighbourhood \(|z| < R\) of the origin, represent a branch of an algebraic function which is not a polynomial. Let \( b \) denote a rational number such that \( 0 < |b| < R \). Let \( S_n = \sum_{\nu=0}^{n} a_{\nu} b^\nu \) \((n = 0, 1, 2, \ldots)\). Denote by \( p_n \) the largest prime divisor in the numerator of \( S_n \).

(i) If \( f(b) \neq 0 \), then \( \limsup_{n \to \infty} p_n = \infty \).

(ii) If \( f(b) \) is an irrational number, then \( \lim_{n \to \infty} p_n = \infty \). (This last statement implies that the limit exists in an extended sense.)

By studying Popken’s proof of this theorem, one sees that the condition that \( f(z) \) is an algebraic function can be somewhat modified without affecting the proof. We illustrate this remark by using Popken’s original proof, but with different hypotheses, to derive a similar result. This new version of Popken’s theorem, combined with a result of Mahler [3, Theorem 1, p. 57], enables us to obtain an interesting consequence about transcendental values of lacunary analytic functions taken at rational points. This last result is an extension of a particular case of the following theorem due to Cohn [2].

**COHN’S THEOREM.** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^{e_k} \) be a lacunary power series with rational coefficients \( a_k = p_k/q_k \) \((k \geq 0)\). Let \( R \) be the radius of convergence of \( f \), \( A_k = \max_{i=0,\ldots,k} |a_i| \), and \( M_k \) the least common multiple of \( q_0, \ldots, q_k \). If

\[
\lim_{k \to \infty} \frac{e_k + \log M_k + \log A_k}{e_k + 1} = 0,
\]

then \( f(b) \) is transcendental for every algebraic number \( b \), with \( 0 < |b| < R \).

This formulation of Cohn’s result is taken from Cijsouw and Tijdeman [1]; indeed, Cijsouw and Tijdeman generalized Cohn’s result to the case where the \( a_k \) are algebraic integers.

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**THEOREM 1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series which is not a polynomial with rational integral coefficients \( a_n \) \((n = 0, 1, 2, \ldots)\) converging in a neighbourhood \( |z| < R \) of the origin. Let \( b \) denote a rational number such that \( 0 < |b| < R \). Let \( S_n = \sum_{\nu=0}^{n} a_\nu b^\nu \) \((n = 0, 1, 2, \ldots)\). Let \( p_n \) denote the largest prime divisor in the numerator of \( S_n \).

(i) If \( f(b) \) is a nonzero algebraic number, then \( \limsup_{n \to \infty} p_n = \infty \).

(ii) If \( f(b) \) is an algebraic irrational number, then \( \lim_{n \to \infty} p_n = \infty \).

**PROOF.** (i) Put \( b = u/v \), where \( u \) and \( v > 0 \) are rational integers. Then

\[
S_n = \sum_{\nu=0}^{n} \frac{a_\nu u^\nu}{v^\nu} = \frac{x_n}{y_n},
\]

with \( y_n = v^n \) and \( x_n \) an integer for \( n \geq 1 \). Denote the prime divisors of \( v \) by \( p_1, p_2, \ldots, p_g \). Now suppose the assertion \( \limsup_{n \to \infty} p_n = \infty \) is false. Then all integers \( y_i, x_i \) have a finite number of prime divisors \( p_1, p_2, \ldots, p_w \) \((w > g)\). Thus

\[
x_i = \pm p_1^{\xi_1} p_2^{\xi_2} \cdots p_w^{\xi_w}, \quad y_i = p_1^{\eta_1} p_2^{\eta_2} \cdots p_2^{\eta_w} \quad (i = 0, 1, 2, \ldots),
\]

where the \( \xi \)'s and \( \eta \)'s are nonnegative integers.

Since \( 0 < |b| < R \), there exists a positive number \( \delta \) so small that \( \omega := (\delta + 1/R)|b| < 1 \). If \( R' \) \((\geq R)\) denotes the radius of convergence of \( \sum a_n z^n \), then

\[
\limsup_{n \to \infty} \sqrt[n]{|a_n|} = \frac{1}{R'}.
\]

Hence, for sufficiently large \( i \),

\[
\left| f(b) - \frac{x_i}{y_i} \right| = \left| \sum_{\nu=i+1}^{\infty} a_\nu b^\nu \right| \leq \sum_{\nu=i+1}^{\infty} \left( \delta + \frac{1}{R} \right)^\nu |b|^\nu = \frac{\omega^{1+i}}{1-\omega}.
\]

Choose a positive number \( k \) so small that \( 1/v^k > \omega \). Then for sufficiently large \( i \),

\[
y_i^{-k} = v^{-ki} > \omega^i/(1-\omega).
\]

Hence, for sufficiently large \( i \),

\[
|f(b) - x_i/y_i| < \omega^i/(1-\omega) < y_i^{-k}.
\]

Since \( f(b) \neq 0 \) is an algebraic number, by Ridout's theorem [5],

\[
f(b) = x_i/y_i \quad \text{for sufficiently large } i.
\]

It follows that \( a_i b^i = S_i - S_{i-1} = 0 \). Hence, \( a_i = 0 \) for sufficiently large \( i \), so \( f(z) \) is necessarily a polynomial, which is a contradiction.

(ii) The proof is similar to (i). We assume the assertion is false. Then there exists an increasing sequence \((n_j)\) such that all numerators of \( S_{n_j} \) can be formed from a finite number of primes. By the same arguments as in (i), using \((n_j)\) instead of \((n)\) and \( S_{n_j} \) rather than \( S_n \), we arrive at the fact that, for sufficiently large \( j \), \( f(b) = x_{n_j}/y_{n_j} \). This contradicts the irrationality of \( f(b) \), and Theorem 1 is proved.

We remark that, with only a slight change in the proof, Theorem 1 is still true if the coefficients \( a_k \) satisfy the Eisenstein condition, i.e. \( \exists N \in \mathbb{N} \) such that \( N^k a_k \in \mathbb{Z} \) \((k = 0, 1, 2, \ldots)\).
3. Transcendental values of lacunary power series. For our application we need the following result of Mahler [3, Theorem 1, p. 57].

Mahler's Theorem. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with rational integral coefficients converging in a neighbourhood \( |z| < R \) of the origin. Suppose \( f(z) \) satisfies the gap condition, that is, there are two infinite sequences of integers, \( \{r_n\} \) and \( \{s_n\} \), satisfying

\[
0 = s_0 < r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \cdots, \quad \lim_{n \to \infty} \frac{s_n}{r_n} = \infty,
\]

such that \( a_h = 0 \) if \( r_n < h < s_n \), but \( a_{r_n} \neq 0, a_{s_n} \neq 0 \) (\( n = 1, 2, 3, \ldots \)). Let \( b \) be an algebraic number satisfying \( |b| < R \). Then \( f(b) \) is algebraic if and only if there exists a positive integer \( N = N(b) \) such that \( P_n(b) = 0 \) for all \( n \geq N \), where

\[
P_n(z) = \sum_{h=s_n}^{r_{n+1}} a_h z^h \quad (n = 0, 1, 2, \ldots).
\]

We are now ready to prove

Theorem 2. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a lacunary power series with properties as in Mahler's Theorem. If \( b \) is a rational number satisfying \( 0 < |b| < R \), then \( f(b) \) is either a rational or a transcendental number.

Proof. Suppose, to the contrary, that \( f(b) \) is an algebraic irrational number. Then by Mahler's Theorem there exist a positive integer \( N = N(b) \) and an increasing sequence of positive integers \( (n_j) \) such that \( s_{n_j} = r_{n_j} \) for all \( j \). Thus the limiting value of the largest prime divisors \( p_n \) of \( S_n \) (as \( n \) tends to infinity) either does not exist, or, if it does, it is never infinite. This contradicts Theorem 1(ii), and our result follows.

We conclude with a few remarks.

1. The case where \( s_n = r_{n+1} \) \( (n = 0, 1, 2, \ldots) \) in Theorem 2 corresponds to a special case of Cohn's Theorem mentioned earlier. In this case \( f(b) \) is necessarily a transcendental number, because, if not, Mahler's Theorem then implies \( P_n(b) = a_{s_n} b^{s_n} = 0 \) for all sufficiently large \( n \), so \( f(z) \) reduces to a polynomial, which is a contradiction.

2. The possibility, in Theorem 2, of \( f(b) \) being a rational number does indeed exist, as shown by the following example. Take for \( n \geq 1 \),

\[
r_{n+1} = s_n + 1, \quad a_{r_{n+1}}/a_{s_n} = a_{s_{n+1}}/a_{s_n} = -1/b.
\]

Then \( P_n(b) = 0 \) for all \( n \geq 1 \), so \( f(b) = a_0 + a_{r_1} b^{r_1} \) is a rational number.

References


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