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STRONGLY π-REGULAR MATRIX SEMIGROUPS

JAN OKNIŃSKI

ABSTRACT. We prove that if S is a strongly π-regular multiplicative subsemigroup of the matrix algebra $M_n(K)$, K being a field, then there exists a chain of ideals $S_1 \lhd \cdots \lhd S_t = S$ such that $t \leq 2^{n+1}$ and any Rees factor semigroup S_i/S_{i-1} is either completely 0-simple or nilpotent of index not exceeding $\prod_{j=0}^{n} \binom{n}{j}$. This sharpens the main result of [4], in particular solving Problem 3.9 from [3].

A semigroup S is said to be strongly π-regular if for any $s \in S$ there exists an integer $k \geq 1$ such that s^k lies in a subgroup of S. Matrix semigroups (i.e. multiplicative subsemigroups of the matrix algebra $M_n(K)$ for some field K and some $n \geq 1$) of this type have been recently the subject of considerable study (for references see [4]). In particular, Putcha proved that any such semigroup admits a chain of ideals $S_1 \lhd \cdots \lhd S_t = S$ with all factor semigroups S_i/S_{i-1} being completely 0-simple or nilpotent [4]. The purpose of this note is to show that, in this case, t cannot exceed 2^{n+1} and that there is a bound (also depending on n only) on indices of nilpotency of the nilpotent factors. The former solves Problem 3.9 from [3]. The way we shall proceed is quite different from that used in [4] where the Zariski closure was the main tool.

If $a \in M_n(K)$, then by $\rho(a)$ we shall denote the rank of a. Further, for any j, $1 \leq j \leq n$, we put $I_j = \{a \in M_n(K) | \rho(a) \leq j\}$. Certainly, I_j is an ideal of the semigroup $M_n(K)$. Let $a \in M_n(K)$. Then, treating a as a linear transformation of an n-dimensional vector space over K with a fixed basis, we shall denote by $\Lambda^j a$ the jth exterior power of a and treat it (in a usual way) as an element of $M_{\binom{n}{j}}(K)$.

We are indebted to Dr. Z. Marciniak for bringing the following well-known result as well as its usefulness for our considerations to our attention.

LEMMA 1 (CF. [1, §5, EXERCISE 11]). *Let $1 \leq j \leq n$. Then $\Lambda^j \colon M_n(K) \to M_{\binom{n}{j}}(K)$ is a semigroup homomorphism. Moreover $\rho(\Lambda^j(a)) = 0$ if $\rho(a) < j$ and $\rho(\Lambda^j(a)) = \binom{\rho(a)}{j}$ if $\rho(a) \geq j$.*

We shall start with an auxiliary result concerning matrices of rank one.

LEMMA 2. *Let $T \subset M_n(K)$ be a set of idempotents of rank one. Then:*
(1) if $(ef)^2 = 0$ for any $e, f \in T$, $e \neq f$, then $|T| \leq 2^n - 1$,
(2) if $(efg)^2 = 0$ for any $e, f, g \in T$, $e \neq g$, then $|T| \leq n$.

PROOF. We will proceed by induction on n. The result is obvious if $n = 1$. Let $n > 1$ and $f, g \in T$, $f \neq g$. Since $\rho(g) = \rho(f) = 1$, the condition $(fg)^2 = 0$ easily implies that $fg = 0$ or $gf = 0$. Thus, in any case, $(1-g)f(1-g)$ is an idempotent.

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Let us fix $e \in T$ and define $\overline{f} = (1-e)f(1-e)$ for $f \in T \backslash \{e\}$, $\overline{T} = \{\overline{f} | f \in T\backslash\{e\}\}$. Then $\overline{T}$ embeds into $M_{n-1}(K)$. Since $\overline{f} = 0$ would imply $ef = f$ or $fe = f$, then by hypothesis $\rho(\overline{f}) = 1$ for any $\overline{f} \in \overline{T}$.

(1) Put $T_1 = \{f \in T | ef = 0\}$, $T_2 = \{f \in T | fe = 0\}$ and $\overline{T}_i = \{\overline{f} | f \in T_i\}$ for $i = 1, 2$. If $f, g \in T_1$, then $(\overline{f}\overline{g})^2 = (fg)^2(1-e)$. Hence $(\overline{f}\overline{g})^2 = 0$ if and only if $f \neq g$. This means that $\overline{T}_1$ satisfies the induction hypothesis and $|\overline{T}_1| = |T_1|$. Since similar arguments can be applied to the set T_2, by the induction argument we then get

$$|T| \leq |T_1| + |T_2| + 1 = |\overline{T}_1| + |\overline{T}_2| + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1.$$

(2) It may be easily checked that for any $\overline{f}, \overline{g} \in \overline{T}$, $f \neq g$, the element $\overline{f}\overline{g}$ must be nilpotent. For example, if $ef = fg = ge = 0$, then $(\overline{f}\overline{g})^2 = (feg)^2 = 0$. Hence, in particular, $\overline{f} = \overline{g}$ implies $f = g$. Thus, as above, the inclusion argument yields $|T| = |\overline{T}| + 1 \leq n$.

It is easy to see that the assumption of (1) in Lemma 2 is essentially weaker than that of (2). In fact, take, for example,

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\} \subset M_2(K).$$

COROLLARY 1. *Let $T \in M_n(K)$ be a set of idempotents of rank j. Then*
(1) *if $\rho((ef)^2) < j$ for any $e, f \in T$, $e \neq f$, then $|T| \leq 2^{\binom{n}{j}} - 1$,*
(2) *if $\rho((efg)^2) < j$ for any $e, f, g \in T$, $e \neq g$, then $|T| \leq \binom{n}{j}$.*

PROOF. Let us consider the set $\Lambda^j(T) \subset M_{\binom{n}{j}}(K)$. Since Λ^j is a homomorphism, then the assumption of (1) (hence, also the assumption of (2)) implies that $|T| = |\Lambda^j(T)|$. Moreover, by Lemma 1, $\Lambda^j(T)$ consists of idempotents of rank one satisfying the hypotheses of Lemma 2(1), (2) accordingly. Thus, the result follows from Lemma 2.

The first part of Corollary 1 solves Problem 3.9 from [**3**]. However, to get a stronger result in our structure theorem we will use the second assertion of Corollary 1.

PROPOSITION 1. *Let $S \subset M_n(K)$ be a strongly π-regular semigroup. Then S has at most 2^n regular $\mathcal{J}$-classes.*

PROOF. Let $1 \leq j \leq n$. Assume that some idempotents $e, f, g \in S$ of rank j are given with e, g lying in distinct $\mathcal{J}$-classes. Suppose that $\rho((efg)^2) = j$. Then, since S is strongly π-regular, there exist $x \in S$ and $y = y^2 \in S$ such that $(efg)^k x = y$ and $\rho(y) = \rho((efg)^k)$ for some $k \geq 1$. Thus $\rho(y) = j$. Now $y = y^2 = (ey)^2$ and so $\rho(eye) \geq \rho(y) = j$ which implies that $\rho(eye) = j$. Since $(eye)^2 = eye$, this yields $eye = e$. Hence $(efg)^k xe = eye = e$ and so $e \in SgS$. Similarly $g \in SeS$, a contradiction. Now, from Corollary 1 it follows that there are at most $\binom{n}{j}$ regular $\mathcal{J}$-classes of S contained in $I_j \backslash I_{j-1}$. This yields the result.

Let us observe that there exists a semigroup $T \subset M_n(K)$ which is strongly π-regular and has exactly 2^n regular $\mathcal{J}$-classes—namely the set of all diagonal idempotents.

PROPOSITION 2. *Let $S \subset M_n(K)$ be a strongly π-regular semigroup. If I is an ideal of S such that S/I is a nil semigroup, then $(S/I)^m = 0$, where $m = \prod_{j=1}^{n} \binom{n}{j}$.*

PROOF. Let j be the least integer such that $S \subset I_j$. Define H as the subsemigroup generated by $S \backslash I$. Let $h = g_1 \cdots g_s$, $g_i \in S \backslash I$. By hypothesis, there exist $y \in S$, $e = e^2 \in S$, $k \geq 1$, such that $h^k y = (g_1 \cdots g_s)^k y = e$ and $\rho(e) = \rho(h^k)$. Suppose that $\rho(e) = j$. Then $\rho(h) = \rho(e)$ and from [**4**, Lemma 4], it follows that $g_1 = eg_1$. Since I is an ideal of S, then we must have $e \notin I$. This contradicts the fact that S/I is nil. Hence $\rho(h^k) = \rho(e) < j$. While $h \in H$ is an arbitrary element, $H/(H \cap I_{j-1})$ is a nil semigroup.

Let us consider the semigroup $\Lambda^j(H) \subset M_{\binom{n}{j}}(K)$. Since, by Lemma 1, $\Lambda^j(I_{j-1}) = 0$, then the first part of the proof implies that $\Lambda^j(H)$ is a nil semigroup. Thus, it is well known that $\Lambda^j(H^{\binom{n}{j}}) = \Lambda^j(H)^{\binom{n}{j}} = 0$ [**2**, Proposition 17.19]. This means that $H^{\binom{n}{j}} \subset I_{j-1}$. Now we have a natural epimorphism $H/(H \cap I_{j-1}) \to S/(I \cup (I_{j-1} \cap S))$ and the latter is also nilpotent of index not exceeding $\binom{n}{j}$. On the other hand

$$(I \cup (I_{j-1} \cap S))/I \simeq (I_{j-1} \cap S)/(I \cap I_{j-1} \cap S) = (I_{j-1} \cap S)/(I_{j-1} \cap I).$$

Putting $\overline{S} = I_{j-1} \cap S$, $\overline{I} = I_{j-1} \cap I$, we get a strongly π-regular semigroup $\overline{S} \subset I_{j-1}$ and we may repeat the above procedure regarding the nil semigroup $\overline{S}/\overline{I}$. Thus, after j such steps we get $(S/I)^{\binom{n}{j}\binom{n}{j-1}\cdots\binom{n}{1}} = 0$. Hence $(S/I)^m = 0$.

Let us notice that Proposition 2 provides in particular an alternative proof of the implication S/I-nil $\Rightarrow$ S/I-nilpotent, which was proved in [**4**].

Now, by Propositions 1 and 2 and the structure theorem for strongly π-regular semigroups with finitely many regular $\mathcal{J}$-classes (cf. [**3**, Lemma 1.9]) we can summarize our results.

THEOREM. *Let $S \subset M_n(K)$ be a strongly π-regular semigroup. Then there exists a chain of ideals $S_1 \lhd \cdots \lhd S_t = S$ such that $t \leq 2^{n+1}$ and all factors S_i/S_{i-1} are completely 0-simple or nilpotent of index not exceeding $m = \prod_{j=1}^{n} \binom{n}{j}$.*

At last, observe that, in view of [**4**], the Theorem establishes the nonexistence of chains of left (right) principal ideals of S with length exceeding $m^{2^{n+1}}$, m as above.

ADDED IN PROOF. Since any periodic semigroup S is strongly π-regular, the Theorem provides a simple proof of the Burnside theorem for semigroups (cf. [**5**]). In fact, to show that S is locally finite it is enough to prove that all S_i/S_{i-1} are locally finite (cf. [**6**, Lemma 3]). While the nilpotent case is obvious, the completely 0-simple case easily follows by Rees' theorem and the Burnside theorem for torsion groups.

REFERENCES

1. N. Bourbaki, *Algèbre*, Chapitre III, Hermann, Paris, 1948.
2. C. Faith, *Algebra. II. Ring theory*, Springer-Verlag, Berlin and New York, 1976.
3. M. Putcha, *On algebraic linear semigroups*, Trans. Amer. Math. Soc. **259** (1980), 457–469.
4. ——, *Matrix semigroups*, Proc. Amer. Math. Soc. **88** (1983), 386–390.
5. R. McNaughton and Y. Zalcstein, *The Burnside problem for semigroups*, J. Algebra **34** (1975), 292–299.
6. J. Okniński, *On regular semigroup rings*, Proc. Roy. Soc. Edinburgh (to appear).

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, 00-901 WARSAW, POLAND