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SELF-DUAL LATTICES FOR MAXIMAL ORDERS IN GROUP ALGEBRAS

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ABSTRACT. Let G be a finite group and V an irreducible $\mathbf{Q}[G]$-module. Let R be a Dedekind domain with quotient field $\mathbf{Q}$ such that $|G|$ is a unit in R. For applications to topology it is of interest to know if V contains a full self-dual $R[G]$-lattice. We show that such lattices always exist for some major classes of finite groups.

Let G be a finite group and let R be a Dedekind domain with quotient field $\mathbf{Q}$ such that $|G|$ is a unit in R. We say that a $\mathbf{Q}[G]$-module V is *balanced* if V contains a full self-dual $R[G]$-lattice. For applications to surgery theory (see [**10**, p. 28–36]) it is of interest to establish criteria for balance.

We show that any irreducible $\mathbf{Q}[G]$-module is balanced when G is p-hyperelementary for an odd prime p, when $|G|$ is odd, or when G is a 2-group. We know of no example of an unbalanced $\mathbf{Q}[G]$-module for any finite group G. Theorem 3, our main criterion for balance, follows easily from standard but deep results in integral representation theory.

I would like to thank Bruce Williams for bringing this problem to my attention.

Preliminaries. For G and R as above, $R[G]$ is a maximal order in $\mathbf{Q}[G]$ by [**9**, Theorem 41.1]. Therefore every left $R[G]$-lattice L is projective, and L is indecomposable if and only if $\mathbf{Q}L$ is an irreducible $\mathbf{Q}[G]$-module [**9**, Corollary 21.5]. The central primitive idempotents e_i $(1 \leq i \leq m)$ of $\mathbf{Q}[G]$ lie in $R[G]$ by [**9**, Theorem 10.5], and any $R[G]$-lattice L decomposes as $L = e_1 L \oplus \cdots \oplus e_m L$. If L and M are isomorphic $R[G]$-lattices, so are $e_i L$ and $e_i M$ for each i. By [**9**, Theorems 11.1 and 18.7] two $R[G]$-lattices L and M belong to the same genus if and only if $\mathbf{Q}L \cong \mathbf{Q}M$. If $H \leq G$ and L is an $R[H]$-lattice, then L^G denotes the induced lattice $R[G] \otimes_{R[H]} L$.

If L is a left $R[G]$-lattice, then L^* denotes the dual (contragredient) left $R[G]$-lattice. If L affords the matrix representation $\rho: G \to \mathrm{GL}(n, R)$, then L^* affords the composition of ρ with the inverse transpose automorphism of $\mathrm{GL}(n, R)$. In particular, $L \cong L^{**}$. For e_i as above, $e_i L^* \cong (e_i L)^*$. If L is an $R[H]$-lattice for some $H \leq G$, then $(L^*)^G \cong (L^G)^*$.

If V is a $\mathbf{Q}[G]$-module, let χ_V denote the character of V. If χ is an irreducible complex character of G, let $m(\chi)$ denote the Schur index of χ over $\mathbf{Q}$, and let $\mathrm{Tr}(\chi)$ denote the sum of the distinct algebraic conjugates of χ. If ψ is any rational-valued character of G, let $p(\psi)$ be the permutation index of ψ—the least integer p such that $p\psi$ is an integral linear combination of permutation characters of G. By [**4**, Theorem 5.21] $p(\psi)$ divides $|G|$.

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The following lemmas contain the key results we need from integral representation theory. If M and N are $R[G]$-lattices, we write $M|N$ to mean that M is isomorphic to a direct summand of N.

LEMMA 1. *Let V be an irreducible $\mathbf{Q}[G]$-module. Let M and N be $R[G]$-lattices such that $\mathbf{Q}M \cong mV$ and $\mathbf{Q}N \cong nV$, with $m < n$. Then $M|N$.*

PROOF. Let $M = M_1 \oplus \cdots \oplus M_m$ and $N = N_1 \oplus \cdots \oplus N_n$ be decompositions of M and N as direct sums of indecomposable lattices. Since M_1, N_1, and N_2 belong to the same genus, we may write $N_1 \oplus N_2 \cong M_1 \oplus L_1$ for an indecomposable $R[G]$-lattice L_1 by [**9**, Corollary 27.3]. Let $M' = M_2 \oplus \cdots \oplus M_m$ and let $N' = L_1 \oplus N_3 \oplus \cdots \oplus N_n$. Then $M \cong M_1 \oplus M'$ and $N \cong M_1 \oplus N'$. By induction on m we may assume $M' \mid N'$. Hence $M \mid N$. ■

We say that a $\mathbf{Q}[G]$-module V is Eichler if no simple component of $\mathrm{End}_{\mathbf{Q}[G]}V$ is a totally definite quaternion algebra, as defined in [**9**, p. 293]. We note that if V is an irreducible $\mathbf{Q}[G]$-module which is not Eichler, and χ is an irreducible complex constituent of χ_V, then $\chi(1) = m(\chi) = 2$. The structure of $G/\mathrm{Ker}\ V$ is severely restricted; see [**9**, p. 344]. We say that an $R[G]$-lattice L is Eichler if $\mathbf{Q}L$ is Eichler.

LEMMA 2. *Let X, M, and N be $R[G]$-lattices. If $X \oplus M \cong X \oplus N$ and M is Eichler, then $M \cong N$.*

PROOF. See [**5**, p. 14]. ■

The main criterion.

THEOREM 3. *Let V be an irreducible $\mathbf{Q}[G]$-module. Suppose $p(\chi_V)$ is odd and V is Eichler. Then V is balanced.*

PROOF. Let $p = p(\chi_V)$. Then $pV \oplus V_1 \cong V_2$ for $\mathbf{Q}[G]$-permutation modules V_1 and V_2. Let L_1 and L_2 be full self-dual $R[G]$-lattices in V_1 and V_2, respectively. Let e be the primitive central idempotent in $\mathbf{Q}[G]$ which corresponds to V. Then eL_1 and eL_2 are full self-dual $R[G]$-lattices in eV_1 and eV_2, respectively. By Lemma 1 we may write $eL_2 = eL_1 \oplus L_0$ for an $R[G]$-lattice L_0 with $\mathbf{Q}L_0 \cong pV$. Taking duals yields $eL_2 \cong eL_1 \oplus L_0^*$. By Lemma 2 we have $L_0 \cong L_0^*$.

Now let M be a fixed $R[G]$-lattice with $\mathbf{Q}M = V$. Let $M_0 = \frac{1}{2}(p-1)(M \oplus M^*)$. By Lemma 1 we may write $L_0 \cong M_0 \oplus M_1$, where $\mathbf{Q}M_1 \cong V$. Since $M_0^* \cong M_0$ and $L_0^* \cong L_0$. we have $L_0 \cong M_0 \oplus M_1^*$. Lemma 2 yields $M_1 \cong M_1^*$. ■

We recall that a group G is called p-hyperelementary if G has a cyclic normal p-complement.

COROLLARY 4. *Suppose G is p-hyperelementary for an odd prime p, $|G|$ is odd, or G is abelian. Then every irreducible $\mathbf{Q}[G]$-module V is balanced.*

PROOF. Suppose first that G is p-hyperelementary for an odd prime p. Let χ be an irreducible complex constituent of χ_V. By [**4**, Theorem 6.15] $\chi(1)$ is odd, so V is Eichler. By [**6**, Definition 1.6 and Proposition 7.2] χ_V has odd permutation index. Hence, V is balanced by Theorem 3. A similar argument works if $|G|$ is odd.

If G is cyclic, then G is p-hyperelementary for any odd prime p, so V is balanced. If G is abelian, then $G/\mathrm{Ker}\ V$ is cyclic, so V is balanced. ■

COROLLARY 5. *Let G be a finite group and V an irreducible $\mathbf{Q}[G]$-module. If V is Eichler and V_H is balanced for every 2-hyperelementary subgroup H of G, then V is balanced.*

PROOF. Let $\mathcal{H}$ be the family of all hyperelementary subgroups of G. By [**4**, Theorem 8.10] we may write

$$1_G = \sum_{H\in\mathcal{H}} a_H 1_H^G - \sum_{H\in\mathcal{H}} b_H 1_H^G,$$

where all the a_H and b_H are nonnegative integers. Then

$$\chi_V = \sum a_H(\chi_V|_H)^G - \sum b_H(\chi_V|_H)^G.$$

Hence,

$$V \oplus \bigoplus_{H\in\mathcal{H}} b_H(V_H)^G \cong \bigoplus_{H\in\mathcal{H}} a_H(V_H)^G.$$

By the hypotheses and Corollary 4, V_H and, hence, $(V_H)^G$ are balanced for all $H \in \mathcal{H}$. Let L_1 and L_2 be full self-dual $R[G]$-lattices in $\bigoplus b_H(V_H)^G$ and $\bigoplus a_H(V_H)^G$, respectively. The argument in the first paragraph of the proof of Theorem 3 shows that V is balanced. ■

REMARKS. When G is 2-hyperelementary and V is an irreducible $\mathbf{Q}[G]$-module, there is a subgroup H of G and a primitive $\mathbf{Q}[H]$-module W such that $V = W^G$. Let $\overline{H} = H/\mathrm{Ker}\, W$. Since every normal abelian subgroup of $\overline{H}$ is cyclic, an application of [**11**, Lemma 2.3] to $\mathbf{O}_2(\overline{H})$ shows that $\overline{H}$ contains a self-centralizing normal cyclic subgroup. Thus the question of whether V is balanced reduces in a sense to a Galois action situation, as in the Brauer-Witt theorem on Schur indices; see [**4**, Theorem 10.7].

We also remark that Corollaries 4 and 5 do not exhaust the applications of Theorem 3. See [**2, 7** and **8**] for more information about permutation indices.

2-groups. We prove a strong form of the balance property for 2-groups.

PROPOSITION 6. *Let G be a 2-group and let V be an irreducible $\mathbf{Q}[G]$-module. Then V contains a full self-dual $\mathbf{Z}[G]$-lattice.*

PROOF. Let G be a 2-group with a faithful irreducible primitive $\mathbf{Q}[G]$-module V. To prove the proposition, it suffices to show that $|G| \leq 2$. Since G has no noncyclic normal abelian subgroup, [**3**, Theorem 5.4.10] shows that G is cyclic, dihedral, semidihedral, or generalized quaternion. Also, $G \neq D_8$ and we may assume $G \neq 1$.

Suppose G is not cyclic and $|G| > 8$. Let $< x >$ be the maximal cyclic subgroup of G and choose $t \in G$ so that $G =< t, x >$. If G is not generalized quaternion, choose t to be an involution. Let $G_0 =< t, x^2 >$. Let λ be a faithful linear character of $< x >$ and let $\lambda(x) = \epsilon$. Then ϵ is a primitive 2^nth root of 1 for some $n \geq 3$. Let $\chi = \lambda^G$ and let $\chi_0 = \chi|_{G_0} = (\lambda|_{<x^2>})^{G_0}$. Then χ and χ_0 are irreducible complex characters of G and G_0, respectively.

The field of values $\mathbf{Q}(\chi_0)$ is contained in $\mathbf{Q}(\epsilon^2)$, while $\chi(x) = \epsilon + \epsilon^{-1}$ or $\epsilon + \epsilon^{2^{n-1}-1}$. Let σ be the unique nonidentity field automorphism of $\mathbf{Q}(\epsilon)$ which fixes ϵ^2. Then $\epsilon^\sigma = \epsilon^{2^{n-1}+1} = -\epsilon$. Hence, $\chi(x)^\sigma = -\chi(x) \neq 0$, so that $\chi^\sigma \neq \chi$, $\chi_0^G = \chi + \chi^\sigma$, and $[\mathbf{Q}(\chi) : \mathbf{Q}(\chi_0)] = 2$. By [**1**, 11.7 and 11.8] we have $m(\chi) = m(\chi_0)$.

Therefore $(m(\chi_0)\mathrm{Tr}\ \chi_0)^G = m(\chi)\mathrm{Tr}(\chi)$. It follows that V is induced from an irreducible $\mathbf{Q}[G_0]$-module, contrary to assumption.

Thus $G = Q_8$ or G is cyclic. In these cases G has a unique faithful irreducible $\mathbf{Q}[G]$-module which is induced from the unique subgroup of G of order 2. Since V is primitive, $|G| = 2$. ∎

FINAL REMARK. The referee has pointed out that it would be more significant to discuss whether a $\mathbf{Q}[G]$-module contains a self-dual $\mathbf{Z}[G]$-lattice rather than a self-dual $R[G]$-lattice. For the application to topology, however, our results on $R[G]$-lattices are significant. They show that, for appropriate G, the computation of relative surgery obstruction groups arising in a certain long exact sequence reduces to the computation of surgery obstruction groups for maximal orders in division algebras.

REFERENCES

1. W. Feit, *Characters of finite groups*, Benjamin, New York, 1967.
2. D. Gluck, *A characterization of generalized permutation characters*, J. Algebra **63** (1980), 541–547.
3. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.
4. I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.
5. H. Jacobinski, *Genera and decomposition of lattices over orders*, Acta Math. **121** (1968), 1–29.
6. T. Y. Lam, *Artin exponent of finite groups*, J. Algebra **9** (1968), 94–119.
7. J. Rasmussen, *Rationally represented and permutation characters of nilpotent groups*, J. Algebra **29** (1974), 504–509.
8. ——, *Artin index of faithful metacyclic groups*, J. Algebra **46** (1977), 511–522.
9. I. Reiner, *Maximal orders*, Academic Press, London, 1975.
10. I. Hambleton, L. Taylor and B. Williams, *An introduction to maps between surgery obstruction groups*, preprint.
11. T. R. Wolf, *Solvable and nilpotent subgroups of* $\mathrm{GL}(n, q^m)$, Canad. J. Math. **34** (1982), 1097–1111.

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