PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 2, February 1985

ACTION OF THE AUTOMORPHISMS OF A SMOOTH DOMAIN IN $\mathbf{C}^n$

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ABSTRACT. A transformation rule relating the Bergman projection and an element of the Lie algebra of $\mathrm{Aut}(\Omega)$ is given, and this is used to give a proof that the action of $\mathrm{Aut}(\Omega)$ extends smoothly to $\bar{\Omega}$.

We let $\Omega \Subset \mathbf{C}^n$ be a bounded domain with C^∞ smooth boundary. $\mathrm{Aut}(\Omega)$ will denote the group of automorphisms (i.e. biholomorphisms $g\colon \Omega \to \Omega$). By a theorem of H. Cartan, $\mathrm{Aut}(\Omega)$ has the structure of a Lie group such that the group action A: $\Omega \times \mathrm{Aut}(\Omega) \to \Omega$ given by $A(z, g) = g(z)$ is real analytic. We let $\mathrm{Aut}_0(\Omega)$ denote the connected component of the identity in $\mathrm{Aut}(\Omega)$.

Let us recall that a domain satisfies Condition R if the Bergman projection P preserves smoothness, i.e. $P(C^\infty(\bar{\Omega})) \subset C^\infty(\bar{\Omega})$ (cf. [3]). The following is known (see [**1, 3**]).

THEOREM 1. *Let $\Omega \Subset \mathbf{C}^n$ be a smoothly bounded domain satisfying Condition R. Then each $g \in \mathrm{Aut}(\Omega)$ extends to a diffeomorphism $g\colon \bar{\Omega} \to \bar{\Omega}$, and the action of $\mathrm{Aut}(\Omega)$ on Ω extends to a smooth action $A\colon \bar{\Omega} \times \mathrm{Aut}(\Omega) \to \bar{\Omega}$.*

Here we give a transformation formula for the elements of the Lie algebra of $\mathrm{Aut}(\Omega)$ and show how this yields a different proof of Theorem 1. We consider the infinitesimal generators of this action. Let $G = \mathrm{Aut}_0(\Omega)$, $\mathfrak{g}$ = Lie algebra of G, and let $\exp\colon \mathfrak{g} \to G$ denote the exponential mapping. Let the mapping from $\mathfrak{g}$ to vector fields on Ω be given by $X \to V_X$, where

$$V_X\phi = \frac{d}{dt}(\phi(\exp(tX)))\Big|_{t=0}. \tag{1}$$

Since V_X is a real vector field, we may write

$$V_X = \sum a_j(z)\frac{\partial}{\partial z_j} + \bar{a}_j(z)\frac{\partial}{\partial \bar{z}_j}.$$

By (1) we see that $V_X\phi$ is holomorphic whenever ϕ is, so each a_j is holomorphic on Ω.

LEMMA. *If $\phi \in C_0^\infty(\Omega)$, then for $V = V_X$ we have*

$$VP\phi + AP\phi = PV\phi + P(A\phi),$$

where $A(z) = \sum_{j=1}^n \partial a_j(z)/\partial z_j$ denotes the divergence of V.

Received by the editors February 27, 1984 and, in revised form, April 10, 1984.
1980 *Mathematics Subject Classification*. Primary 32M05; Secondary 32H10, 32D15.

PROOF. Let us write $g_t = \exp(tX)$. By the transformation law for the Bergman projection under g_t, we have

$$J_t(z)(P\phi)(g_t(z)) = P(J_t(z)\phi(g_t(z))),$$

where J_t denotes the Jacobian determinant of g_t. Differentiating this expression with respect to t and setting $t = 0$, we have

$$\left(\frac{dJ_t}{dt}\Big|_{t=0}\right)P\phi + V_X(P\phi) = P\left[\left(\frac{d}{dt}J_t\Big|_{t=0}\right)\phi + V\phi\right].$$

The Lemma follows from the observation that

$$g_t(z) = z + t(a_1(z),\dots,a_n(z)) + O(t^2).$$

Thus

$$\frac{\partial g_t^i(z)}{\partial z_j} = \delta_{ij} + t\frac{\partial a_i(z)}{\partial z_j} + O(t^2).$$

We find that

$$\frac{d}{dt}J_t\Big|_{t=0} = A(z)$$

since

$$\frac{d}{dt}\det(I + tB)\big|_{t=0} = \operatorname{trace}(B).$$

THEOREM 2. *Let $\Omega \Subset \mathbf{C}^n$ be smoothly bounded and satisfy Condition R. Then for all $X \in \mathfrak{g}$ the corresponding infinitesimal transformation V_X extends smoothly to $\overline{\Omega}$.*

PROOF. We show that for $p \in \partial\Omega$ and any multi-index α, $\partial^{|\alpha|}a_k/\partial z^\alpha$ is bounded on $\Omega \cap \{|z - p| < \varepsilon\}$.

By a well-known result of S. Bell [1], $P(C_0^\infty(\overline{\Omega})) = P(C^\infty(\overline{\Omega}))$. By the closed graph theorem, P: $C_0^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ is continuous. Thus $P(C_0^\infty(\Omega))$ is dense in $P(C_0^\infty(\overline{\Omega}))$, which is evidently equal to $C^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$. Thus there exist $\phi_0, \phi_1,\dots,\phi_n \in C_0^\infty(\Omega)$ such that

$$P\phi_0 = 1 + O\left(|z - p|^2\right), \qquad P\phi_j = z_j - p_j + O\left(|z - p|^2\right)$$

for $j = 1,2,\dots,n$. We apply the Lemma to $\phi = \phi_0 + c_1\phi_1 + \cdots + c_n\phi_n$ with constant c_j and use

$$VP\phi + AP\phi \in P\left(C_0^\infty(\Omega)\right) \subset P\left(C^\infty(\overline{\Omega})\right), \tag{3}$$

where the inclusion follows from Condition R.

For fixed $z \in \Omega$ with $|z - p| < \varepsilon$, we may choose c with $|c| = 1$ such that $VP\phi(z) = 0$. By (3), then, $A(z)$ is bounded near p. Now (3) gives

$$\sum_{j=1}^{n} a_j\left(c_j + O\left(|z - p|\right)\right) \in L^\infty. \tag{4}$$

Since we may choose $c_j = \bar{a}_j/|a|$, we see that $a_j \in L^\infty$.

Now we proceed by induction, assuming that $\partial^{|\beta|}a_k/\partial z^\beta$ and $\partial^{|\beta|}A/\partial z^\beta$ are bounded for $|\beta| < |\alpha|$. Differentiating (3), we obtain

$$(5) \qquad \sum_{j=1}^{n} \frac{\partial^{|\alpha|}a_j}{\partial z^\alpha}\big(c_j + O(|z-p|)\big) + \frac{\partial^{|\alpha|}A}{\partial z^\alpha}\big(1 + O(|z-p|)\big) \in L^\infty.$$

Since (5) holds for all choices of c_j, we see that all the terms on the left-hand side are bounded, which completes the proof.

PROOF OF THEOREM 1. We consider first the action $A:\overline{\Omega} \times \mathrm{Aut}_0(\Omega) \to \overline{\Omega}$ of the connected component of the identity. Theorem 1 in this case follows from Theorem 2 by Lie's Theorem, since the action of G on $\overline{\Omega}$ may be obtained by integrating the vector fields V_X on $\overline{\Omega}$. (See Chapter 9 of [**4**].)

For the general case it is sufficient to show that if H is a connected component of $\mathrm{Aut}(\Omega)$, then $A:\overline{\Omega} \times H \to \overline{\Omega}$ is smooth. This follows since $H = g\,\mathrm{Aut}_0(\Omega)$ for some $g \in \mathrm{Aut}(\Omega)$, and g extends smoothly to $\overline{\Omega}$ by [**3**].

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