MARTIN COMPACTIFICATIONS
AND QUASICONFORMAL MAPPINGS

SHIGEO SEGAWA AND TOSIMASA TADA

Dedicated to Professor Yukio Kusunoki on the occasion of his 60th birthday.

Abstract. It is shown that there exists a quasiconformal mapping $T$ of a Riemann surface $R_1$ onto another $R_2$ such that $T$ cannot be extended to a homeomorphism of the Martin compactification $R^*_1$ of $R_1$ onto that $R^*_2$ of $R_2$.

It has long been asked whether a quasiconformal mapping between two Riemann surfaces can be extended to a homeomorphism between two Martin compactifications of the given surfaces. In literature this interesting problem was first explicitly stated in an expository paper by Royden [5]. Since then the problem seems to have been open and here we wish to settle the question in the negative. Namely,

Theorem. There exists a quasiconformal mapping $T$ of a Riemann surface $R_1$ onto another $R_2$ such that $T$ cannot be extended to a homeomorphism of the Martin compactification $R^*_1$ of $R_1$ onto that $R^*_2$ of $R_2$.

In passing we remark that the Theorem is invalid if the Martin compactification is replaced by the Royden compactification (cf., e.g., Sario and Nakai [6]). Concerning the other compactifications such as Kuramochi and Wiener compactifications the question seems to be entirely open (cf., e.g., Constantinescu and Cornea [3]).

The proof of the Theorem is achieved by giving examples of $(R_1, R_2, T)$. Actually we can choose $R_1$ and $R_2$ to be plane regions of rather simple characters which were considered by Ancona [1]. In §1 an example of Ancona, mentioned above, is stated as the first Lemma. A minimality criterion also given by Ancona [2] is then stated as the second Lemma (in §2). Based upon these two preliminary results the construction of $(R_1, R_2, T)$ is carried over in §3.

1. We start by fixing notation. Let $\alpha$ and $\beta$ be positive numbers with $\alpha + \beta < \pi$. Using decreasing sequences $\{t_n\}_1^\infty$ and $\{t'_n\}_1^\infty$ in $(0, 1/2]$ with $t_n = \lim t'_n = 0$, we consider the radial slits

$$T_n = \{re^{i\alpha}; t_{2n} \leq r \leq t_{2n-1}\}, \quad T'_n = \{re^{-i\alpha}; t'_{2n} \leq r \leq t'_{2n-1}\}$$
and the region
\[ \Omega_0 = \{ re^{i\theta}; 0 < r < 2, |\theta| < \alpha + \beta \} - \bigcup_{n=1}^{\infty} (T_n \cup T'_n). \]

The following surprisingly interesting fact is due to Ancona [1].

**Lemma.** If \( 2\alpha < \beta \), then there exist sequences \( \{t_n\}, \{t'_n\} \) in \( (0, 1/2) \) such that the Martin boundary of \( \Omega_0 \) lying over \( z = 0 \) contains two distinct minimal points \( \xi \) and \( \xi' \), and
\[ \lim_{n \to \infty} \frac{1}{2} t_n = \xi, \quad \lim_{n \to \infty} \frac{1}{2} t'_n = \xi'. \]

2. We need another result of Ancona [2] on a criterion of minimality. Let \( \Omega \) be a plane region and \( x \) a relative boundary point of \( \Omega \). Then the above cited criterion is

**Lemma.** If there exists an open disk \( B(y, r) \) of center \( y \) and radius \( r \) with the property that \( B(y, r) \subset \Omega \) and \( x \notin \partial B(y, r) \), the relative boundary of \( B(y, r) \), then there exists a minimal Martin boundary point \( \xi \) lying over \( x \) with
\[ \lim_{t \to 0} (x + t(y - x)) = \xi. \]

3. For any \( \alpha \) and \( \beta \) with \( 0 < 2\alpha < \beta \) and \( \alpha + \beta < \pi \), consider a \( C^1 \) function \( f(\theta) \) on \( [-\beta - \alpha, \alpha + \beta] \) such that \( f(\pm(\alpha + \beta)) = \pm \pi, f(\pm \alpha) = \pm \pi/2, f(0) = 0 \), and
\[ C^{-1} \leq f'(\theta) = \frac{d}{d\theta} f(\theta) \leq C \]
for a constant \( C \) with \( C > 1 \). Let \( \Omega_0 \) be a region for which the Lemma in §1 is valid, and let \( \psi \) be the mapping from \( \Omega_0 \) to \( \psi(\Omega_0) \) defined by
\[ \psi(re^{i\theta}) = re^{if(\theta)}. \]
Since the dilatation \( K_\psi(z) \) of \( \psi \) at \( z = re^{i\theta} \) satisfies
\[ K_\psi(z) = \frac{|1 + f'(\theta)| + |1 - f'(\theta)|}{|1 + f'(\theta)| - |1 - f'(\theta)|} \leq C, \]
\( \psi \) is a quasiconformal mapping from \( \Omega_0 \) to \( \psi(\Omega_0) \) (cf., e.g., Lehto and Virtanen [4]).

Observe that the mapping \( \psi \) fixes sequences \( \{t_n\}_1^\infty \) and \( \{t'_n\}_1^\infty \) obtained in the Lemma in §1 termwise. Since the region \( \psi(\Omega_0) \) contains the disk \( B(1, 1) \) and the interval \( (0, 1] \) contains both the sequences \( \{\psi(t_n)\}_1^\infty \) and \( \{\psi(t'_n)\}_1^\infty \), the Lemma in §2 assures that \( \{\psi(t_n)\} \) and \( \{\psi(t'_n)\} \) converge to an identical minimal point in the Martin compactification of \( \psi(\Omega_0) \). On the other hand, by the Lemma in §1, limits of \( \{t_n\} \) and \( \{t'_n\} \) are distinct. Therefore \( \psi \) cannot be extended to a homeomorphism between Martin compactifications of \( \Omega_0 \) and \( \psi(\Omega_0) \).

**References**


Department of Mathematics, Daido Institute of Technology, Daido, Minami, Nagoya 457, Japan