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MARTIN COMPACTIFICATIONS AND QUASICONFORMAL MAPPINGS

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Dedicated to Professor Yukio Kusunoki on the occasion of his 60th birthday.

ABSTRACT. It is shown that there exists a quasiconformal mapping T of a Riemann surface R_1 onto another R_2 such that T cannot be extended to a homeomorphism of the Martin compactification R_1^* of R_1 onto that R_2^* of R_2.

It has long been asked whether a quasiconformal mapping between two Riemann surfaces can be extended to a homeomorphism between two Martin compactifications of the given surfaces. In literature this interesting problem was first explicitly stated in an expository paper by Royden [**5**]. Since then the problem seems to have been open and here we wish to settle the question in the *negative*. Namely,

THEOREM. *There exists a quasiconformal mapping T of a Riemann surface R_1 onto another R_2 such that T cannot be extended to a homeomorphism of the Martin compactification R_1^* of R_1 onto that R_2^* of R_2.*

In passing we remark that the Theorem is invalid if the Martin compactification is replaced by the Royden compactification (cf., e.g., Sario and Nakai [**6**]). Concerning the other compactifications such as Kuramochi and Wiener compactifications the question seems to be entirely open (cf., e.g., Constantinescu and Cornea [**3**]).

The proof of the Theorem is achieved by giving examples of (R_1, R_2, T). Actually we can choose R_1 and R_2 to be *plane regions* of rather simple characters which were considered by Ancona [**1**]. In §1 an example of Ancona, mentioned above, is stated as the first Lemma. A minimality criterion also given by Ancona [**2**] is then stated as the second Lemma (in §2). Based upon these two preliminary results the construction of (R_1, R_2, T) is carried over in §3.

1. We start by fixing notation. Let α and β be positive numbers with $\alpha + \beta < \pi$. Using decreasing sequences $\{t_n\}_1^\infty$ and $\{t_n'\}_1^\infty$ in $(0, 1/2\,]$ with $\lim t_n = \lim t_n' = 0$, we consider the radial slits

$$T_n = \left\{re^{i\alpha};\, t_{2n} \leqslant r \leqslant t_{2n-1}\right\}, \qquad T_n' = \left\{re^{-i\alpha};\, t_{2n}' \leqslant r \leqslant t_{2n-1}'\right\}$$

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and the region

$$\Omega_0 = \{re^{i\theta}; 0 < r < 2, |\theta| < \alpha + \beta\} - \bigcup_{n=1}^{\infty} (T_n \cup T_n').$$

The following surprisingly interesting fact is due to Ancona [**1**].

LEMMA. *If $2\alpha < \beta$, then there exist sequences $\{t_n\}$, $\{t_n'\}$ in $(0, 1/2]$ such that the Martin boundary of Ω_0 lying over $z = 0$ contains two distinct minimal points ζ and ζ', and*

$$\lim_{n\to\infty} \frac{1}{2} t_n = \zeta, \qquad \lim_{n\to\infty} \frac{1}{2} t_n' = \zeta'.$$

2. We need another result of Ancona [**2**] on a criterion of minimality. Let Ω be a plane region and x a relative boundary point of Ω. Then the above cited criterion is

LEMMA. *If there exists an open disk $B(y, r)$ of center y and radius r with the property that $B(y, r) \subset \Omega$ and $x \in \partial B(y, r)$, the relative boundary of $B(y, r)$, then there exists a minimal Martin boundary point ζ lying over x with*

$$\lim_{t\to 0} (x + t(y - x)) = \zeta.$$

3. For any α and β with $0 < 2\alpha < \beta$ and $\alpha + \beta < \pi$, consider a C^1 function $f(\theta)$ on $[-\beta - \alpha, \alpha + \beta]$ such that $f(\pm(\alpha + \beta)) = \pm\pi, f(\pm\alpha) = \pm\pi/2, f(0) = 0$, and

$$C^{-1} \leqslant f'(\theta) = \frac{d}{d\theta} f(\theta) \leqslant C$$

for a constant C with $C > 1$. Let Ω_0 be a region for which the Lemma in §1 is valid, and let ψ be the mapping from Ω_0 to $\psi(\Omega_0)$ defined by

$$\psi(re^{i\theta}) = re^{if(\theta)}.$$

Since the dilatation $K_\psi(z)$ of ψ at $z = re^{i\theta}$ satisfies

$$K_\psi(z) = \frac{|1 + f'(\theta)| + |1 - f'(\theta)|}{|1 + f'(\theta)| - |1 - f'(\theta)|} \leqslant C,$$

ψ is a quasiconformal mapping from Ω_0 to $\psi(\Omega_0)$ (cf., e.g., Lehto and Virtanen [**4**]).

Observe that the mapping ψ fixes sequences $\{t_n\}_1^\infty$ and $\{t_n'\}_1^\infty$ obtained in the Lemma in §1 termwise. Since the region $\psi(\Omega_0)$ contains the disk $B(1, 1)$ and the interval $(0, 1]$ contains both the sequences $\{\psi(t_n)\}_1^\infty$ and $\{\psi(t_n')\}_1^\infty$, the Lemma in §2 assures that $\{\psi(t_n)\}$ and $\{\psi(t_n')\}$ converge to an identical minimal point in the Martin compactification of $\psi(\Omega_0)$. On the other hand, by the Lemma in §1, limits of $\{t_n\}$ and $\{t_n'\}$ are distinct. Therefore ψ cannot be extended to a homeomorphism between Martin compactifications of Ω_0 and $\psi(\Omega_0)$.

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