PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 2, February 1985

LOCAL UNCERTAINTY INEQUALITIES FOR FOURIER SERIES

JOHN F. PRICE AND PAUL C. RACKI

ABSTRACT. Necessary and sufficient conditions are given on α, β and t for there to exist a constant K such that

$$\left(\sum_{n\in E} |\hat{f}(n)|^2\right)^{1/2} \leqslant K|E|^{\alpha}\left\|f|x|^{\beta}\right\|_t$$

for all $f \in L^1(T^d)$ and finite $E \subset Z^d$.

1. Introduction. The classical uncertainty principle inequality [2] states that

$$\||x|f\|_2\||y|F\|_2 \geqslant (d/4\pi)\|f\|_2^2 \tag{1.1}$$

for all functions $f \in L^2(\mathbf{R}^d)$, where the Fourier transform F is defined by

$$F(y) = \int_{\mathbf{R}^d} f(x)\exp(-2\pi i x.y)\,dx \quad \text{for } y = (y_1,\ldots,y_d) \in \mathbf{R}^d.$$

The natural generalization of this to a product $\mathbf{T}^d = (\mathbf{R}/2\pi\mathbf{Z})^d$ of circle groups fails since $\sum_{n\in\mathbf{Z}^d} |n|^2|\hat{f}(n)|^2 = 0$ for all constant functions $f \in L^2(\mathbf{T}^d)$. (Here and below,

$$\hat{f}(n) = (2\pi)^{-d}\int_{\mathbf{T}^d} f(x)\exp(-in.x)\,dx \quad \text{for } n = (n_1,\ldots,n_d) \in \mathbf{Z}^d.\Big)$$

Recently, local versions of (1.1) have been developed and applied in quantum physics [3, 4]. Roughly speaking, they assert that if a function is condensed, then not only is its Fourier transform broad, but it cannot be "too" localized. Some of these inequalities involve general L^p-norms and powers of $|x|$. For example, in [4] it is shown that, given $t \in [1,\infty]$ and $\beta \in \mathbf{R}$, there exists a constant K such that

$$\left(\int_E |F(y)|^2\,dy\right)^{1/2} \leqslant Km(E)^{\beta - d/t^{\#}}\left\|f|x|^{\beta}\right\|_t$$

for all $f \in L^2(\mathbf{R}^d)$ and measurable $E \subset \mathbf{R}^d$, with $m(E) < \infty$ if and only if $d/t^{\#} < \beta < d/t'$ or $(t,\beta) = (1,0)$ or $(2,0)$, where $t' = t/(t-1)$ and $t^{\#} = 2t/(t-2)$. Furthermore, no other power of $m(E)$ apart from $\beta - d/t^{\#}$ is possible. (Global extensions of (1.1) are given in [1].)

Here we establish the following analogous result for $\mathbf{T}^d$.

Received by the editors December 12, 1983 and, in revised form, February 27, 1984.
1980 *Mathematics Subject Classification*. Primary 42B05; Secondary 26D05, 42A05.

THEOREM. *Given $t \in [1, \infty]$ and $\alpha, \beta \in \mathbf{R}$, there exists constant K such that for all functions $f \in L^1(\mathbf{T}^d)$,*

$$\left(\sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leqslant K|E|^{\alpha} \left\| f|x|^{\beta} \right\|_t \tag{1.2}$$

for all finite $E \subset \mathbf{Z}^d$ if and only if α, β, t satisfy the following conditions (see Figure 1):

(i) $\beta < d/t'$ *if* $1 < t \leqslant \infty$, *otherwise* $\beta \leqslant d/t'$ *if* $t = 1$;

(ii) $\alpha \geqslant \max\{0, -1/t^{\#}\}$ *if* $\beta \leqslant 0$ *and* $1 \leqslant t \leqslant 2$, *or* $\beta < d/t^{\#}$ *and* $t < 2$;

(iii) $d\alpha \geqslant \beta - d/t^{\#}$ *if* $\max\{0, d/t^{\#}\} < \beta < d/t'$.

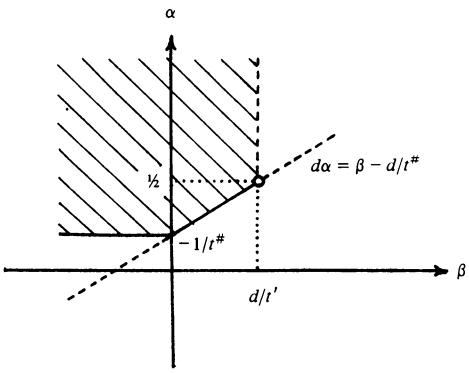

FIGURE 1a

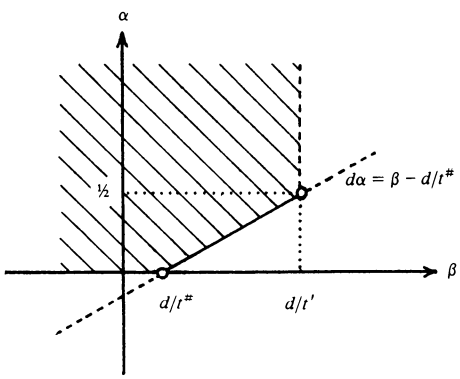

FIGURE 1b

The shaded area of Figure 1(a) is the region of validity of the inequality for $1 \leqslant t \leqslant 2$ with the boundary $\beta = d/t'$ ($\alpha \geqslant \frac{1}{2}$) being included for $t = 1$. When $2 < t \leqslant \infty$ the region of validity is the shaded area of Figure 1(b).

2. Proof of sufficiency. Assuming the conditions of the theorem hold, let $N \subset \mathbf{T}^d$, $N' = \mathbf{T}^d - N$ be its complement, and f be a function in $L^2(\mathbf{T}^d)$. We thus have $f = f\chi_N + f\chi_{N'}$, where χ_N is the characteristic function on N, so that $\hat{f} = (f\chi_N)\hat{} + (f\chi_{N'})\hat{}$. Hence,

$$\left(\sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leqslant \left(\sum_{n \in E} |(f\chi_n)\hat{}(n)|^2 \right)^{1/2} + \left(\sum_{n \in E} |(f\chi_{N'})\hat{}(n)|^2 \right)^{1/2} \tag{2.1}$$

by Minkowski's inequality; we shall estimate the two quantities separately. First, using Hölder's inequality,

(2.2)

$$\left(\sum_{n \in E} |(f\chi_N)\hat{}(n)|^2 \right)^{1/2} \leqslant \max\left\{ |(f\chi_N)\hat{}(n)| : n \in E \right\} |E|^{1/2} \leqslant \|f\chi_N\|_1 |E|^{1/2}$$

$$= \left\| f\chi_N |x|^{\beta} |x|^{-\beta} \right\|_1 |E|^{1/2} \leqslant \left\| f\chi_N |x|^{\beta} \right\|_t \left\| \chi_N |x|^{-\beta} \right\|_{t'} |E|^{1/2}.$$

Considering the second term of (2.1), we have

$$\begin{aligned}(2.3)\quad \Big(\sum_{n\in E}\big|(f\chi'_N)\hat{}(n)\big|^2\Big)^{1/2} &\leqslant \Big(\sum_{n\in Z^d}\big|(f\chi_{N'})\hat{}(n)\big|^2\Big)^{1/2} = \|f\chi_{N'}\|_2\\ &= \big\|f\chi_{N'}|x|^{\beta}|x|^{-\beta}\big\|_2 \leqslant \big\|f\chi_{N'}|x|^{\beta}\big\|_{2r}\big\|\chi_{N'}|x|^{-\beta}\big\|_{2r'}\\ &= \big\|f\chi_{N'}|x|^{\beta}\big\|_t\big\|\chi_{N'}|x|^{-\beta}\big\|_{t^\#},\end{aligned}$$

where we used Hölder's inequality with $r = t/2$ and assumed $t > 2$.

If we assume that $t \in [1, 2]$ we estimate the second term as follows:

$$\begin{aligned}(2.4)\quad \Big(\sum_{n\in E}\big|(f\chi_{N'})\hat{}(n)\big|^2\Big)^{1/2} &= \Big(\sum_{n\in E}\big|(f\chi_{N'})\hat{}(n)\cdot 1\big|^2\Big)^{1/2}\\ &\leqslant \Big(\sum_{n\in E}\big|(f\chi_{N'})\hat{}(n)\big|^{2r}\Big)^{1/2r}\Big(\sum_{n\in E}1^{2r'}\Big)^{1/2r'}\\ &= \Big(\sum_{n\in E}\big|(f\chi_{N'})\hat{}(n)\big|^{t'}\Big)^{1/t'}|E|^{-1/t^\#}\\ &\leqslant \Big(\sum_{n\in Z^d}\big|(f\chi_{N'})\hat{}(n)\big|^{t'}\Big)^{1/t'}|E|^{-1/t^\#}\\ &\leqslant \|f\chi_{N'}\|_t|E|^{-1/t^\#} = \big\|f\chi_{N'}|x|^{\beta}|x|^{-\beta}\big\|_t|E|^{-1/t^\#}\\ &\leqslant \big\|f\chi_{N'}|x|^{\beta}\big\|_t\big\|\chi_{N'}|x|^{-\beta}\big\|_\infty|E|^{-1/t^\#},\end{aligned}$$

where we used Hölder's inequality with $r = t'/2$, followed by Hausdorff-Young and a final application of Hölder's inequality.

Substituting (2.2)–(2.4) into (2.1), we obtain

$$(2.5)\qquad \Big(\sum_{n\in E}|\hat{f}(n)|^2\Big)^{1/2} \leqslant K\big\|f|x|^{\beta}\big\|_t,$$

where $K = K_1 + K_2$, with

$$K_1 = \big\|\chi_N|x|^{-\beta}\big\|_{t'}|E|^{1/2},$$

$$K_2 = \begin{cases}\big\|\chi_{N'}|x|^{-\beta}\big\|_\infty|E|^{-1/t^\#} & \text{for } 1 \leqslant t \leqslant 2,\\ \big\|\chi_{N'}|x|^{-\beta}\big\|_{t^\#} & \text{for } 2 < t \leqslant \infty.\end{cases}$$

Letting $N = \{x \in \mathbf{T}^d\colon |x| < a\}$ for some a, we now evaluate the above norms. First,

$$\begin{aligned}(2.6)\qquad \big\|\chi_N|x|^{-\beta}\big\|_{t'}^{t'} &= \int_{|x|\leqslant a}|x|^{-\beta t'}\,dx = W_d\int_0^a r^{-\beta t'}r^{d-1}\,dr\\ &= \big(W_d/(d-\beta t')\big)a^{d-\beta t'},\end{aligned}$$

as $\beta < d/t'$, where $W_d = 2\pi^{d/2}/\Gamma(d/2)$. Second

$$(2.7)\quad \left\|\chi_{N'}|x|^{-\beta}\right\|_{t^\#}^{t^\#} = \int_{a\leqslant|x|\leqslant\pi} |x|^{-\beta t^\#}\,dx = W_d\int_a^\pi r^{-\beta t^\#+d-1}\,dr$$

$$= W_d(\beta t^\# - d)^{-1}(a^{d-\beta t^\#} - \pi^{d-\beta t^\#}) \leqslant W_d(\beta t^\# - d)^{-1}a^{d-\beta t^\#},$$

as $\beta > \max\{0, d/t^\#\}$. Thirdly if $\beta > 0$,

$$(2.8)\qquad \left\|\chi_{N'}|x|^{-\beta}\right\|_\infty = \sup\left\{|x|^{-\beta}: a \leqslant |x| \leqslant \pi\right\} = a^{-\beta}.$$

Now, letting $a = |E|^{-1/d}$ and substituting (2.6)–(2.8) into (2.5), we obtain

$$\left(\sum_{n\in E} |\hat{f}(n)|^2\right)^{1/2} \leqslant K|E|^{\beta/d-1/t^\#}\left\|f|x|^\beta\right\|_t,$$

with $K = K_1 + K_2$, where $K_1 = (W_d/(d-\beta t'))^{1/t'}$ and

$$K_2 = \begin{cases} 1 & \text{for } 1 \leqslant t \leqslant 2, \\ \left(W_d/(\beta t^\# - d)\right)^{1/t^\#} & \text{for } 2 < t \leqslant \infty, \end{cases}$$

provided $\max\{0, d/t^\#\} < \beta < d/t'$.

Finally, if $\beta \leqslant 0$, $\|\chi_{N'}|x|^{-\beta}\|_\infty = \pi$; hence, $K_2 = \pi|E|^{-1/t^\#}$ and

$$K_1 = \left(W_d/(d-\beta t')\right)^{1/t'} a^{d/t'-\beta}|E|^{1/2} = \left(W_d/(d-\beta t')\right)^{1/t'}|E|^{-1/t^\#}$$

upon letting $a = |E|^{-1/(d-\beta t')}$ for $t \neq 1$. If $t = 1$,

$$K_1 = \left\|\chi_N|x|^{-\beta}\right\|_\infty |E|^{1/2} = a^{-\beta}|E|^{1/2} = |E|^{-1/t^\#}$$

upon letting $a = 1$. Thus

$$\left(\sum_{n\in E} |\hat{f}(n)|^2\right)^{1/2} \leqslant K|E|^{-1/t^\#}\left\|f|x|^\beta\right\|_t$$

for $\beta \leqslant 0$ and $1 \leqslant t \leqslant 2$ with $K = K_1 + K_2$, where $K_2 = \pi$ and

$$K_1 = \begin{cases} 1 & \text{for } t = 1, \\ \left(W_d/(d-\beta t')\right)^{1/t'} & \text{for } 1 < t \leqslant 2. \end{cases}$$

We thus have the inequality holding along the line segment in the (α, β)-plane given by $d\alpha = \beta - d/t^\#$ for $\max\{0, d/t^\#\} < \beta < d/t'$. When $t = 1$ we also have it holding at the endpoint $\beta = d/t' = 0$ since

$$\left(\sum_{n\in E} |\hat{f}(n)|^2\right)^{1/2} \leqslant |E|^{1/2}\|\hat{f}\|_\infty \leqslant |E|^{1/2}\|f\|_1.$$

These results may be extended as follows: If

$$\left(\sum_{n\in E} |\hat{f}(n)|^2\right)^{1/2} \leqslant K|E|^\alpha\left\|f|x|^\beta\right\|_t$$

holds for given (α, β, t), then it holds for (α', β', t') with K replaced by $K\|\,|x|^{\beta-\beta'}\|_\infty$, where $\alpha' \geqslant \alpha$, $\beta' \leqslant \beta$ *and* $t' \geqslant t$. To see this, notice that $|E|^{\alpha'} \geqslant |E|^\alpha$ and $\|f|x|^\beta\|_t \leqslant \|\,|x|^{\beta-\beta'}\|_\infty\|f|x|^{\beta'}\|_{t'}$.

This leaves only the case $\alpha = 0$ and $t \geqslant 2$; it can be established as follows:

$$\|\hat{f}\|_2 = \|f\|_2 = \left\|f|x|^{\beta}|x|^{-\beta}\right\|_2 \leqslant \left\|f|x|^{\beta}\right\|_t \left\||x|^{-\beta}\right\|_{t^\#}$$

with

$$\left\||x|^{-\beta}\right\|_{t^\#}^{t^\#} = W_d \int_0^{\pi} r^{-\beta t^\# + d - 1}\, dr < \infty \quad \text{for } \beta < d/t^\# \ (t > 2)$$

and

$$\left\||x|^{-\beta}\right\|_{t^\#} = \sup\left\{|x|^{-\beta} : x \in \mathbf{T}^d\right\} < \infty \quad \text{for } \beta \leqslant d/t^\# = 0 \ (t = 2).$$

We have thus covered the stated region, and the sufficiency of the conditions is established.

REMARK. In the interior of the region an alternate value of the constant is $K = \| |x|^{-\beta}\|_{2t/(t+2\alpha t-2)}$. To see this, notice that

$$\begin{aligned}
\left(\sum_{n\in E} |\hat{f}(n)|^2\right)^{1/2} &\leqslant |E|^{1/2r'}\left(\sum_{n\in Z^d} |\hat{f}(n)|^{2r}\right)^{1/2r} \\
&\leqslant |E|^{1/2r'}\left\|f|x|^{\beta}|x|^{-\beta}\right\|_{(2r)'} \\
&\leqslant |E|^{1/2r'}\left\||x|^{-\beta}\right\|_{(2r)'^p}\left\|f|x|^{\beta}\right\|_{(2r)'^{p'}} \\
&= |E|^{\alpha}\left\||x|^{-\beta}\right\|_{2t/(t+2\alpha t-2)}\left\|f|x|^{\beta}\right\|_t.
\end{aligned}$$

3. Proof of necessity. The necessity of the conditions is obtained by ruling out the remaining regions beginning with $\alpha \leqslant 0$. Firstly, we can not have $\alpha < 0$, since then $|E|^{\alpha} \to 0$ as $|E| \to \infty$. Secondly, considering the case $\alpha = 0$, $\beta = d/t^\#$ with $t > 2$, the function $f \in L^1(T^d)$ defined by

$$f(x) = \begin{cases} |x|^{-d/2}\left|\log|x|\right|^{-1/2} & \text{for } |x| \leqslant 1/2, \\ 0 & \text{otherwise} \end{cases}$$

provides a counterexample, since

$$\|f\|_2^2 = W_d \int_0^{1/2} r^{-1}\log^{-1} r\, dr = \infty,$$

while

$$\left\|f|x|^{d/t^\#}\right\|_t^t = W_d \int_0^{1/2} r^{-1}|\log r|^{-t/2}\, dr < \infty \quad \text{if } t > 2.$$

This also rules out the region $\alpha = 0$, $\beta > d/t^\#$ for $t > 2$. Thirdly, consider the region given by $\alpha = 0$, $\beta < d/t^\#$ with $1 \leqslant t < 2$. Here define $f(x) = ||x|-1|^{-1/2}$ for $1/2 \leqslant |x| \leqslant 3/2$ and 0 otherwise, so that

$$\|f\|_2^2 = W_d \int_{1/2}^{3/2} |r-1|^{-1} r^{d-1}\, dr = \infty,$$

while

$$\left\|f|x|^{\beta}\right\|_t^t = W_d \int_{1/2}^{3/2} |r-1|^{-t/2} r^{\beta t + d - 1}\, dr < \infty \quad \text{for } t < 2.$$

Finally, the region given by $\alpha = 0$, $\beta > d/t^{\#} = 0$ for $t = 2$ is ruled out by $f(x) = |x|^{-d/2}$ since

$$\|f\|_2^2 = W_d \int_0^{\pi} r^{-1}\, dr = \infty,$$

while

$$\left\| f|x|^{\beta} \right\|_2 = W_d \int_0^{\pi} r^{-1+2\beta}\, dr < \infty,$$

which completes the $\alpha \leqslant 0$ case.

Now consider the boundary $\beta \geqslant d/t'$; here the function

$$f_\varepsilon(x) = \begin{cases} |x|^{-d}\left|\log|x|\right|^{-1} & \text{for } 0 < \varepsilon \leqslant |x| \leqslant \frac{1}{2}, \\ 0 & \text{otherwise} \end{cases}$$

provides the required counterexample as $\varepsilon \to 0$. For assume that $E = \{0\}$; then

$$\left(\sum_{n \in E} |\hat{f}_\varepsilon(n)|^2 \right)^{1/2} = |\hat{f}_\varepsilon(0)| = \left| W_d \int_\varepsilon^{1/2} r^{-1} \log^{-1} r\, dr \right| \to \infty \quad \text{as } \varepsilon \to 0,$$

while

$$\left\| f_\varepsilon |x|^{\beta} \right\|_t^t = W_d \int_\varepsilon^{1/2} r^{\beta t - dt + d - 1} |\log r|^{-t} r\, dr \leqslant \text{constant}$$

for all $\varepsilon > 0$ if $\beta t - dt + d - 1 \leqslant -1$ and $t > 1$ or $\beta t - dt + d - 1 > -1$ and $t = 1$. Hence, there is a contradiction for all α if $\beta \geqslant d/t'$ and $t > 1$ or $\beta > d/t'$ and $t = 1$.

Consider now the region $d\alpha < \beta - d/t^{\#}$ for $\max\{0, d/t^{\#}\} \leqslant \beta < d/t'$. Define

$$f_N(x) = \begin{cases} 1 & \text{if } x \in \square_N, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\square_N = \left\{ x = (x_1, \ldots, x_d) \in T^d \colon |x_1| \leqslant 1/N, \ldots, |x_d| \leqslant 1/N \right\}$$

and N is a positive integer. Now

$$\begin{aligned} \hat{f}_N(n) = \hat{f}_N(n_1, \ldots, n_d) &= \int_{\square_N} e^{-in_1x_1} \cdots e^{-in_dx_d}\, dx_1 \cdots dx_d \\ &= 2^d n_1^{-1} \sin n_1/N \cdots n_d^{-1} \sin n_d/N. \end{aligned}$$

Thus, letting $E = \{n = (n_1, \ldots, n_d) \in Z^d \colon |n_1| \leqslant N, \ldots, |n_d| \leqslant N\}$, we have

$$\begin{aligned} \left(\sum_{n \in E} |\hat{f}_N(n)|^2 \right)^{1/2} &= 2^d \left(\sum_{|n_1| \leqslant N} \frac{\sin^2 n_1/N}{n_1^2} \cdots \sum_{|n_d| \leqslant N} \frac{\sin^2 n_d/N}{n_d^2} \right)^{1/2} \\ &\sim 1/N^{d/2} \quad \text{as} N \to \infty \end{aligned}$$

since

$$\lim_{N \to \infty} \sum_{|n| \leqslant N} \left(\frac{n}{N} \right)^{-2} \sin^2\left(\frac{n}{N} \right) N^{-1} = \int_{-1}^{1} x^{-2} \sin^2 x\, dx,$$

while

$$\begin{aligned}|E|^{\alpha}\left\|f_N|x|^{\beta}\right\|_t &= N^{d\alpha}\left(\int_{\square_N}|x|^{\beta t}\,dx\right)^{1/t}\\ &\leqslant N^{d\alpha}\left(\int_{|x|\leqslant d^{1/2}/N}|x|^{\beta t}\,dx\right)^{1/t}\\ &= \left(W_d/(\beta t+d)\right)^{1/t}d^{(\beta/2)+(d/2t)}N^{d\alpha-\beta-d/t},\end{aligned}$$

assuming $\beta > d/t^{\#} > -d/t$. We thus have a contradiction if $-d/2 > d/\alpha - \beta - d/t$, that is, if $d\alpha < \beta - d/t^{\#}$, as required.

Finally, define $g_N(x) = f_N(x - \mathbf{1})$, f_N as above, and $\mathbf{1} = (1,\ldots,1)$, for $N > 1$ being an integer. With E as above we have

$$\left(\sum_{n\in E}|\hat{g}_N(n)|^2\right)^{1/2} \sim N^{-d/2} \quad \text{as } N \to \infty$$

since $|\hat{g}_N(n)| = |\hat{f}_N(n)|$. Therefore consider, for $\beta < 0$,

$$\begin{aligned}|E|^{\alpha}\left\|g_N|x|^{\beta}\right\|_t &= N^{d\alpha}\left(\int_{\square_N-\mathbf{1}}|x|^{\beta t}\,dx\right)^{1/t} = N^{d\alpha}\left(\int_{\square_N}|y+1|^{\beta t}\,dy\right)^{1/t}\\ &\leqslant \left(\frac{1}{2}\right)^{\beta}N^{d\alpha}\left(\int_{\square_N}dy\right)^{1/t} = 2^{d/t-\beta}N^{d\alpha}N^{-d/t}.\end{aligned}$$

We thus obtain a contradiction as $N \to \infty$ if $-d/2 > d\alpha - d/t$, that is, $\alpha < -1/t^{\#}$ for $\beta < 0$.

All the required regions are now eliminated and necessity is established which completes the proof of the Theorem.

References

1. M. G. Cowling and J. F. Price, *Bandwidth versus time concentration: the Heisenberg-Pauli-Weyl inequality*, SIAM J. Math. Anal. **15** (1984), 151–165.

2. D. Dym and H. P. McKean, *Fourier series and integrals*, Academic Press, New York, 1972.

3. W. G. Faris, *Inequalities and uncertainty principles*, J. Math. Phys. **19** (1978), 461–466.

4. J. F. Price, *Inequalities and local uncertainty principles*, J. Math. Phys. **27** (1983), 1711–1714.

School of Mathematics, University of New South Wales, Kensington, New South Wales 2033, Australia