LOCAL UNCERTAINTY INEQUALITIES FOR FOURIER SERIES

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Abstract. Necessary and sufficient conditions are given on \( \alpha, \beta \) and \( t \) for there to exist a constant \( K \) such that

\[
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K|E|^{\alpha} \|f|^{\beta}\|,
\]

for all \( f \in L^1(T^d) \) and finite \( E \subseteq \mathbb{Z}^d \).

1. Introduction. The classical uncertainty principle inequality [2] states that

\[
|\langle \hat{f}, f \rangle | \leq \frac{d}{4\pi} \|f\|_2^2
\]

for all functions \( f \in L^2(\mathbb{R}^d) \), where the Fourier transform \( \hat{f} \) is defined by

\[
\hat{f}(y) = \int_{\mathbb{R}^d} f(x) \exp(-2\pi i x \cdot y) \, dx \quad \text{for} \quad y = (y_1, \ldots, y_d) \in \mathbb{R}^d.
\]

The natural generalization of this to a product \( T^d = (\mathbb{R}/2\pi \mathbb{Z})^d \) of circle groups fails since \( \sum_{n \in \mathbb{Z}^d} |n|^2 |\hat{f}(n)|^2 = 0 \) for all constant functions \( f \in L^2(T^d) \). (Here and below,

\[
\hat{f}(n) = (2\pi)^{-d} \int_{T^d} f(x) \exp(-in \cdot x) \, dx \quad \text{for} \quad n = (n_1, \ldots, n_d) \in \mathbb{Z}^d.
\]

Recently, local versions of (1.1) have been developed and applied in quantum physics [3, 4]. Roughly speaking, they assert that if a function is condensed, then not only is its Fourier transform broad, but it cannot be "too" localized. Some of these inequalities involve general \( L^p \)-norms and powers of \( |x| \). For example, in [4] it is shown that, given \( t \in [1, \infty) \) and \( \beta \in \mathbb{R} \), there exists a constant \( K \) such that

\[
\left( \int_E |F(y)|^2 \, dy \right)^{1/2} \leq Km(E)^{\beta - d/t^*} \|f|^{\beta}\|
\]

for all \( f \in L^2(\mathbb{R}^d) \) and measurable \( E \subseteq \mathbb{R}^d \), with \( m(E) < \infty \) if and only if \( d/t^* < \beta < d/t' \) or \( (t, \beta) = (1, 0) \) or \( (2, 0) \), where \( t' = t/(t - 1) \) and \( t^* = 2t/(t - 2) \). Furthermore, no other power of \( m(E) \) apart from \( \beta - d/t^* \) is possible. (Global extensions of (1.1) are given in [1].)

Here we establish the following analogous result for \( T^d \).

Received by the editors December 12, 1983 and, in revised form, February 27, 1984.

1980 Mathematics Subject Classification. Primary 42B05; Secondary 26D05, 42A05.
**Theorem.** Given $t \in [1, \infty]$ and $\alpha, \beta \in \mathbb{R}$, there exists constant $K$ such that for all functions $f \in L^1(T^d)$,

\[
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K |E|^{\alpha} \|f|^{\beta}\|_t,
\]

for all finite $E \subset \mathbb{Z}^d$ if and only if $\alpha, \beta, t$ satisfy the following conditions (see Figure 1):

(i) $\beta < d/t^*$ if $1 < t < \infty$, otherwise $\beta \leq d/t^*$ if $t = 1$;

(ii) $\alpha \geq \max\{0, -1/t^*\}$ if $\beta < 0$ and $1 < t < 2$, or $\beta < d/t^*$ and $t < 2$;

(iii) $d\alpha \geq \beta - d/t^*$ if $\max\{0, d/t^*\} < \beta < d/t^*$.

![Figure 1a](image1a.png)

![Figure 1b](image1b.png)

**Figure 1a**

The shaded area of Figure 1(a) is the region of validity of the inequality for $1 < t < 2$ with the boundary $\beta = d/t^*$ ($\alpha \geq \frac{1}{2}$) being included for $t = 1$. When $2 < t < \infty$ the region of validity is the shaded area of Figure 1(b).

**2. Proof of sufficiency.** Assuming the conditions of the theorem hold, let $N \subset T^d$, $N' = T^d - N$ be its complement, and $f$ be a function in $L^2(T^d)$. We thus have $f = f\chi_N + f\chi_{N'}$, where $\chi_N$ is the characteristic function on $N$, so that $\hat{f} = (f\chi_N) + (f\chi_{N'})$. Hence,

\[
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq \left( \sum_{n \in E} |(f\chi_N)^*(n)|^2 \right)^{1/2} + \left( \sum_{n \in E} |(f\chi_{N'})^*(n)|^2 \right)^{1/2}
\]

by Minkowski's inequality; we shall estimate the two quantities separately. First, using Hölder's inequality,

\[
\left( \sum_{n \in E} |(f\chi_N)^*(n)|^2 \right)^{1/2} \leq \max\{|(f\chi_N)^*(n)| : n \in E\} |E|^{1/2} \leq \|f\chi_N\|_1 |E|^{1/2}
\]

\[
= \|f\chi_N|^{\beta}|x|^{-\beta}\|_1 |E|^{1/2} \leq \|f\chi_N|\|_1 |x|^{\beta}\|_1 \chi_N|x|^{-\beta}\|_1 |E|^{1/2}.
\]
Considering the second term of (2.1), we have

\[
\left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right|^2 \right)^{1/2} \leq \left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right|^2 \right)^{1/2} = \| f x_n \|_2
\]

\[
= \left\| f x_n \right\|_2 \left\| x \right\|_2 \leq \left\| f x_n \right\|_2 \left\| x \right\|_2 \leq \left\| f x_n \right\|_2 \left\| x \right\|_2
\]

where we used Hölder's inequality with \( r = t/2 \) and assumed \( t > 2 \).

If we assume that \( t \in [1,2] \) we estimate the second term as follows:

\[
\left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right|^2 \right)^{1/2} = \left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right|^2 \right)^{1/2} \leq \left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right|^2 \right)^{1/2} \left( \sum_{n \in E} 1^{2r'} \right)^{1/2}
\]

\[
= \left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right| \right)^{1/r} \left| E \right|^{-1/r}
\]

\[
\leq \left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right| \right)^{1/r} \left| E \right|^{-1/r}
\]

\[
\leq \left( \sum_{n \in E} \left| \left( f x_n \right) \hat{(n)} \right| \right)^{1/r} \left| E \right|^{-1/r}
\]

\[
= \left\| f x_n \right\|_2 \left\| x \right\|_2 \leq \left\| f x_n \right\|_2 \left\| x \right\|_2\left\| x \right\|_2
\]

where we used Hölder's inequality with \( r = t'/2 \), followed by Hausdorff-Young and a final application of Hölder's inequality.

Substituting (2.2)–(2.4) into (2.1), we obtain

\[
\left( \sum_{n \in E} \left| \hat{(n)} \right|^2 \right)^{1/2} \leq K \left\| f x \right\|_r,
\]

where \( K = K_1 + K_2 \), with

\[
K_1 = \left\| x_n \right\|_{r,\left| E \right|}^{1/2},
\]

\[
K_2 = \left\{ \begin{array}{ll}
\left\| x_n \right\|_{r,\left| E \right|}^{1/2} & \text{for } 1 \leq t \leq 2, \\
\left\| x_n \right\|_{r,\left| E \right|}^{1/2} & \text{for } 2 < t \leq \infty.
\end{array} \right.
\]

Letting \( N = \{ x \in T^d : |x| < a \} \) for some \( a \), we now evaluate the above norms.

First,

\[
\left\| x_n \right\|_{r,\left| E \right|}^{1/2} = \int_{|x|<a} |x|^{-\beta t'} dx = W_d \int_0^a r^{-\beta t'} r^{d-1} dr
\]

\[
= \left( W_d / (d - \beta t') \right) a^{d-\beta t'},
\]
as $\beta < d/t'$, where $W_d = 2\pi d^{d/2}/\Gamma(d/2)$. Second

\begin{equation}
\|X_N| |x|^{-\beta}\|_a^a = \int_{a \leq |x| \leq \pi} |x|^{-\beta} dx = W_d \int_a^\pi r^{-\beta + d - 1} dr
\end{equation}

as $\beta < \max(0, d/t')$. Thirdly if $\beta > 0$,

\begin{equation}
\|X_N| |x|^{-\beta}\|_\infty = \sup \{ |x|^{-\beta} : a \leq |x| \leq \pi \} = a^{-\beta}.
\end{equation}

Now, letting $a = |E|^{-1/d}$ and substituting (2.6)-(2.8) into (2.5), we obtain

\begin{equation}
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K |E|^{-\beta/d - 1/\beta} \|f| \|_{\beta},
\end{equation}

with $K = K_1 + K_2$, where $K_1 = (W_d/(d - \beta t'))^{1/r'}$ and

\begin{equation}
K_2 = \begin{cases} 
1 & \text{for } 1 \leq t \leq 2, \\
(W_d/(\beta t' - d))^{1/r'} & \text{for } 2 < t < \infty,
\end{cases}
\end{equation}

provided $\max(0, d/t') < \beta < d/t'$.

Finally, if $\beta \leq 0$, $\|X_N| |x|^{-\beta}\|_\infty = \pi$; hence, $K_2 = \pi |E|^{-1/\beta}$ and

\begin{equation}
K_1 = (W_d/(d - \beta t'))^{1/r'} a^{d/r' - \beta} |E|^{1/2} = (W_d/(d - \beta t'))^{1/r'} |E|^{-1/\beta}
\end{equation}

upon letting $a = |E|^{-1/(d - \beta t')}$ for $t \neq 1$. If $t = 1$,

\begin{equation}
K_1 = \|X_N| |x|^{-\beta}\|_\infty |E|^{1/2} = a^{-\beta} |E|^{1/2} = |E|^{-1/\beta}
\end{equation}

upon letting $a = 1$. Thus

\begin{equation}
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K |E|^{-1/\beta} \|f| \|_{\beta},
\end{equation}

for $\beta \leq 0$ and $1 \leq t \leq 2$ with $K = K_1 + K_2$, where $K_2 = \pi$ and

\begin{equation}
K_1 = \begin{cases} 
1 & \text{for } t = 1, \\
(W_d/(d - \beta t'))^{1/r'} & \text{for } 1 < t \leq 2.
\end{cases}
\end{equation}

We thus have the inequality holding along the line segment in the $(\alpha, \beta)$-plane given by $d\alpha = \beta - d/t^\#$ for $\max(0, d/t^\#) < \beta < d/t'$. When $t = 1$ we also have it holding at the endpoint $\beta = d/t' = 0$ since

\begin{equation}
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq |E|^{1/2} \|f| \|_{\infty} \leq |E|^{1/2} \|f| \|_1.
\end{equation}

These results may be extended as follows: If

\begin{equation}
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K |E|^{\alpha} \|f| \|_{\beta},
\end{equation}

holds for given $(\alpha, \beta, t)$, then it holds for $(\alpha', \beta', t')$ with $K$ replaced by $K \| |x|^{\beta - \beta'}\|_\infty$, where $\alpha' \geq \alpha, \beta' \leq \beta and t' > t$. To see this, notice that $|E|^{\alpha'} \geq |E|^{\alpha}$ and $\|f| \|_{\beta, t} \leq \| |x|^{\beta - \beta'}\|_\infty \|f| \|_{\beta'}, t'.

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This leaves only the case $\alpha = 0$ and $t \geq 2$; it can be established as follows:

$$
\|\hat{f}\|_2 = \|f\|_2 = \|f|\beta|x|^{-\beta}\|_2 \leq \|f|\beta\|_\|\cdot\|^\beta\|_\|\cdot\|^\beta,
$$

with

$$
\|\cdot\|^\beta = W_d \int_0^\infty r^{-\beta \alpha + d - 1} \, dr < \infty \quad \text{for } \beta < d/t^# \quad (t > 2)
$$

and

$$
\|\cdot\|^\beta = \sup \{|\cdot|^{-\beta} : x \in T^d\} < \infty \quad \text{for } \beta \leq d/t^# = 0 \quad (t = 2).
$$

We have thus covered the stated region, and the sufficiency of the conditions is established.

**Remark.** In the interior of the region an alternate value of the constant is $K = \|\cdot\|^\beta\|_{2t/(t+2a(-2))}$. To see this, notice that

$$
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq |E|^{1/2r'} \left( \sum_{n \in Z^d} |\hat{f}(n)|^2 \right)^{1/2r'}
$$

$$
\leq |E|^{1/2r'} \|f|\beta\| (2r)^{1/2r'} \|f|\beta\|_{(2r)^{1/2r'}} \|f|\beta\|_{(2r)^{1/2r'}}
$$

$$
= |E|^\alpha \|\cdot\|^\beta \|_{2t/(t+2a(-2))} \|f|\beta\|.
$$

### 3. Proof of necessity

The necessity of the conditions is obtained by ruling out the remaining regions beginning with $\alpha \leq 0$. Firstly, we can not have $\alpha < 0$, since then $|E|^\alpha \rightarrow 0$ as $|E| \rightarrow \infty$. Secondly, considering the case $\alpha = 0$, $\beta = d/t^#$ with $t > 2$, the function $f \in L^1(T^d)$ defined by

$$
f(x) = \begin{cases} 
|x|^{d/2} \log|x|^{-1/2} & \text{for } |x| \leq 1/2, \\
0 & \text{otherwise}
\end{cases}
$$

provides a counterexample, since

$$
\|f\|^2 = W_d \int_0^{1/2} r^{-1} \log^{-1} r \, dr = \infty,
$$

while

$$
\|f|\beta|^{d/\alpha}\|_{t} = W_d \int_0^{1/2} r^{-1} \log^{-1/2} r \, dr < \infty \quad \text{if } t > 2.
$$

This also rules out the region $\alpha = 0$, $\beta > d/t^#$ for $t > 2$. Thirdly, consider the region given by $\alpha = 0$, $\beta < d/t^#$ with $1 \leq t < 2$. Here define $f(x) = |x|^{-1/2}$ for $1/2 \leq |x| \leq 3/2$ and 0 otherwise, so that

$$
\|f\|^2 = W_d \int_{1/2}^{3/2} |r - 1|^{-1} r^{d-1} \, dr = \infty,
$$

while

$$
\|f|\beta|^{t}\|_{t} = W_d \int_{1/2}^{3/2} |r - 1|^{-1/2} r^{t+d-1} \, dr < \infty \quad \text{for } t < 2.
$$
Finally, the region given by \( \alpha = 0, \beta > d/t^* = 0 \) for \( t = 2 \) is ruled out by \( f(x) = |x|^{-d/2} \) since
\[
\|f\|_2^2 = W_d \int_0^\infty r^{-1} dr = \infty,
\]
while
\[
\|f|x|^\beta\|_2 = W_d \int_0^\infty r^{-1+2\beta} dr < \infty,
\]
which completes the \( \alpha \leq 0 \) case.

Now consider the boundary \( \beta \geq d/t^* \); here the function
\[
f_\varepsilon(x) = \begin{cases} |x|^{-d}|\log|x||^{-1} & \text{for } 0 < \varepsilon \leq |x| \leq \frac{1}{\varepsilon}, \\ 0 & \text{otherwise} \end{cases}
\]
provides the required counterexample as \( \varepsilon \to 0 \). For assume that \( E = \{0\} \); then
\[
\left( \sum_{n \in E} |\hat{f}_\varepsilon(n)|^2 \right)^{1/2} = |\hat{f}_\varepsilon(0)| = W_d \int_\varepsilon^{1/2} r^{-1} \log^{-1} r dr \to \infty \quad \text{as } \varepsilon \to 0,
\]
while
\[
\|f_\varepsilon|x|^\beta\|_r = W_d \int_\varepsilon^{1/2} r^{\beta - dt + d - 1} |\log r|^{-t} r dr \leq \text{constant}
\]
for all \( \varepsilon > 0 \) if \( \beta t - dt + d - 1 \leq -1 \) and \( t > 1 \) or \( \beta t - dt + d - 1 > -1 \) and \( t = 1 \). Hence, there is a contradiction for all \( \alpha \) if \( \beta \geq d/t^* \) and \( t > 1 \) or \( \beta > d/t^* \) and \( t = 1 \).

Consider now the region \( d\alpha < \beta - d/t^* \) for \( \max\{0, d/t^*\} < \beta < d/t^* \). Define
\[
f_N(x) = \begin{cases} 1 & \text{if } x \in \Box_N, \\ 0 & \text{otherwise}, \end{cases}
\]
where
\[
\Box_N = \{ x = (x_1, \ldots, x_d) \in T^d: |x_1| \leq 1/N, \ldots, |x_d| \leq 1/N \}
\]
and \( N \) is a positive integer. Now
\[
\hat{f}_N(n) = \hat{f}_N(n_1, \ldots, n_d) = \int_{\Box_N} e^{-in_1 x_1} \cdots e^{-in_d x_d} dx_1 \cdots dx_d
\]
\[
= 2^d n_1^{-1} \sin n_1/N \cdots n_d^{-1} \sin n_d/N.
\]
Thus, letting \( E = \{n = (n_1, \ldots, n_d) \in Z^d: |n_1| \leq N, \ldots, |n_d| \leq N\} \), we have
\[
\left( \sum_{n \in E} |\hat{f}_N(n)|^2 \right)^{1/2} = 2^d \left( \sum_{|n_1| \leq N} \frac{\sin^2 n_1/N}{n_1^2} \cdots \sum_{|n_d| \leq N} \frac{\sin^2 n_d/N}{n_d^2} \right)^{1/2}
\]
\[
\sim 1/N^{d/2} \quad \text{as } N \to \infty
\]
since
\[
\lim_{N \to \infty} \sum_{|n| \leq N} \left( \frac{n}{N} \right)^2 \sin^2 \left( \frac{n}{N} \right) N^{-1} = \int_{-1}^{1} x^{-2} \sin^2 x dx,
\]
while

\[ |E|^\alpha \| f_N |x\|^\beta \|_r = N^{d_\alpha} \left( \int_{\Omega_N} |x|^{\beta_\ell} \, dx \right)^{1/t} \]

\[ \leq N^{d_\alpha} \left( \int_{|x| \leq d^{1/2}/N} |x|^{\beta_\ell} \, dx \right)^{1/t} \]

\[ = (W_d/(\beta_\ell + d))^{1/t} d^{(\beta/2) + (d/2t)} N^{d_\alpha - \beta - d/t}, \]

assuming \( \beta > d/t^2 > -d/t \). We thus have a contradiction if \(-d/2 > d/\alpha - \beta - d/t\), that is, if \( d/\alpha < \beta - d/t^2 \), as required.

Finally, define \( g_N(x) = f_N(x - 1) \), \( f_N \) as above, and \( 1 = (1, \ldots, 1) \), for \( N > 1 \) being an integer. With \( E \) as above we have

\[ \left( \sum_{n \in E} |\hat{g}_N(n)|^2 \right)^{1/2} \sim N^{-d/2} \quad \text{as } N \to \infty \]

since \( |\hat{g}_N(n)| = |\hat{f}_N(n)| \). Therefore consider, for \( \beta < 0 \),

\[ |E|^\alpha \| g_N |x\|^\beta \|_r = N^{d_\alpha} \left( \int_{\Omega_N} |x|^{\beta_\ell} \, dx \right)^{1/t} = N^{d_\alpha} \left( \int_{\Omega_N} |y + 1|^{\beta_\ell} \, dy \right)^{1/t} \]

\[ \leq \left( \frac{1}{2} \right)^{\beta} N^{d_\alpha} \left( \int_{\Omega_N} dy \right)^{1/t} = 2^{d/t} - \beta N^{d_\alpha} N^{-d/t}. \]

We thus obtain a contradiction as \( N \to \infty \) if \(-d/2 > d/\alpha - d/t\), that is, \( \alpha < -1/t^2 \) for \( \beta < 0 \).

All the required regions are now eliminated and necessity is established which completes the proof of the Theorem.

REFERENCES


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