INTEGRABILITY OF POWER SERIES

BABU RAM

Abstract. The object of this paper is to obtain generalization of certain results of Jain [3] and Woyczynski [5] concerning integrability of power series.

1. Introduction. A nondecreasing continuous real-valued function $\Phi$ defined on the nonnegative half line and vanishing only at the origin will be called an Orlicz function (OF). A function $\Phi \in \text{OF}$ is said to satisfy $\Delta_\alpha$ ($\alpha > 0$) condition for large $u$ if there are constants $c > 0$ and $u_0 \geq 0$ such that $\Phi(cu) \leq c\Phi(u)$, $u \geq u_0$. A convex Orlicz function $\Phi$ satisfying the conditions $\lim_{u \to 0} \frac{\Phi(u)}{\Phi(u)/u} = 0$ and $\lim_{u \to \infty} \frac{\Phi(u)}{\Phi(u)/u} = \infty$ is called a Young function (YF). Function $\Phi$ belongs to YF if and only if it admits a representation $\Phi(u) = \int_0^u \phi(t) \, dt$, where $\phi(t)$, $t \geq 0$, is positive, vanishing only at the origin, continuous on the right, nondecreasing and $\lim_{t \to \infty} \phi(t) = \infty$. Then $\Phi(u)/u \leq \phi(u) < \Phi(2u)/u$. Let $\mathcal{M}$ be the class of Orlicz functions which satisfy the following condition:

(\mathcal{M}) There exists a convex function $\Lambda$, $\lambda > 1$, and $0 < \alpha < 1$, such that the inequality $\Lambda(u) \leq \Phi(u) \leq \lambda \Lambda(u)$ holds for all $u$.

Let $L_\Phi(X, \mu)$, where $\Phi \in \Delta_\alpha$, be the Orlicz space, that is, the set of all complex-valued measurable functions $f$ on a measure space $(X, \mu)$ such that the modular $\int_X |f|^\Phi \, d\mu$ is finite. By Hardy-Orlicz space $H_\Phi$ we mean a closed subspace of $L_\Phi((0, 2\pi), dx)$ spanned over trigonometric polynomials of the form $f(t) = \sum_{n=0}^{N} a_n \cos^n t$.

A sequence $(a_n)$ of nonnegative numbers is said to be quasi monotone if for some $a > 0$, $a_{n+1} \leq a_n(1 + a/n)$ or equivalently $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

Let $\Psi(x)$ be a nondecreasing positive function such that $\frac{\Psi(x)}{x^{1-\delta}}$ decreases as $x$ increases in $(0, a)$, where $a > 0$ and $\delta$ is a small constant depending on $\Psi$. Then $\Psi'(x)$ exists [2] almost everywhere and $\Psi'(x) \geq 0$.

2. Results. We prove the following:

Theorem. Let $\Phi \in \Delta_\alpha \cap \mathcal{M} \cap \text{YF}$ and $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $0 \leq x < 1$. If $(a_n)$ is a quasi monotone sequence such that

\begin{equation}
\sum_{k=n}^{\infty} |a_k - a_{k+1}| \leq K a_n,
\end{equation}

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then the following four statements are equivalent:

\begin{align}
(2.1) \quad \frac{\Phi(|f(x)|)}{\Psi(1-x)} & \in L(0,1); \\
(2.2) \quad \frac{\Phi(|f(e^{ix})|)}{\Psi(x)} & \in L(0, \pi); \\
(2.3) \quad \sum_{n=1}^{\infty} \frac{\Phi(na_n)}{n^2\Psi(1/n)} & < \infty; \\
(2.4) \quad \sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2\Psi(1/n)} & < \infty.
\end{align}

In the special case of decreasing sequence \((a_n)\) and \(\Psi \equiv 1\) the above result was obtained by Woyczynski [5] and in the case \(\Psi(x) = x^{-\gamma}, 0 \leq \gamma < 1,\) by Jain [3]. It may also be mentioned here that the author [4] also obtained some results of this type for an increasing, positive and convex function \(\Phi\) with no restriction on \(a_n\).

3. Lemmas. The proof of our theorem is based upon the following lemmas, which are generalizations of the lemmas of Woyczynski [5]. The proof of Lemma 1 also generalizes a classical Hardy’s inequality (see [6, 9.16]). Let us write \(F(x) = \int_0^x f(t) \, dt\).

**Lemma 1.** Let \(\Phi \in \mathcal{M}, a \geq 0.\) Then

\[
\int_0^a \frac{\Phi(F(x)/x)}{\Psi(x)} \, dx \leq K(a) \int_0^a \frac{\Phi(f(x))}{\Psi(x)} \, dx.
\]

**Proof.** By virtue of the assumption and of the Jensen’s inequality there exists an \(\alpha, 0 < \alpha < 1,\) such that

\[
\Phi^\alpha(F(x)/x) \leq \lambda \left( \int_0^x \Phi^\alpha(f(t)) x^{-1} \, dt \right),
\]

whence

\[
(3.1) \quad \int_0^a \frac{\Phi(F(x)/x)}{\Psi(x)} \, dx \leq \lambda^{1/\alpha} \int_0^a \left[ \int_0^x \frac{\Phi^\alpha(f(t)) x^{-1} \, dt}{\Psi(x)} \right]^{1/\alpha} \, dx.
\]

We consider the integral on the right-hand side of (3.1). We write

\[
H(x) = \int_0^x \Phi^\alpha(f(t)) \, dt.
\]

Then the integral under consideration becomes

\[
\int_0^a \left( \frac{H(x)/x}{\Psi(x)} \right)^{1/\alpha} \, dx.
\]
Integration by parts yields
\[
\int_0^a \frac{H(x)/x}{\Psi(x)} \, dx = \frac{a(H(a)/a)^{1/a}}{\Psi(a)} - \int_0^a \frac{H(x)/x}{\alpha \Psi(x)} \, dx - \int_0^a \frac{H(x)/x}{\alpha \Psi(x)} \, dx + \int_0^a \frac{H(x)^{1/a} \Psi'(x)}{x^{1/a-1} \Psi(x)^2} \, dx.
\]

Thus, by the above, and by Hölder’s inequality
\[(3.2)\]
\[
\alpha \left( \frac{1}{\alpha} - 1 \right) \int_0^a \frac{H(x)/x}{\Psi(x)} \, dx \leq \int_0^a \frac{H(x)/x}{\Psi(x)} \, dx - \int_0^a \frac{H(x)/x}{\alpha \Psi(x)} \, dx \leq \left( \int_0^a \frac{\Phi(f(x))}{\Psi(x)} \, dx \right)^{1-a} \left( \int_0^a \frac{H(x)^{1/a} x^{-1/a}}{\Psi(x)} \, dx \right)^{1-a}.
\]

Dividing both sides by the last factor on the right, we get
\[
(1 - \alpha) \left( \int_0^a \frac{H(x)/x}{\Psi(x)} \, dx \right)^{\alpha} \leq \left( \int_0^a \frac{\Phi(f(x))}{\Psi(x)} \, dx \right)^{\alpha}
\]
whence
\[(3.3)\]
\[
\int_0^a \frac{H(x)/x}{\Psi(x)} \, dx \leq K(\alpha) \int_0^a \frac{\Phi(f(x))}{\Psi(x)} \, dx.
\]

The inequalities (3.1) and (3.3) then yield
\[
\int_0^a \frac{\Phi(F(x)/x)}{\Psi(x)} \, dx \leq K(\alpha) \int_0^a \frac{\Phi(xf(x))}{\Psi(x)} \, dx.
\]

**Lemmas.** Let \( \Phi \in \Delta_a \cap YF \) and \( a \geq 0 \). If \( f(x) \) is a nonnegative function, then
\[
\int_0^a \frac{\Phi(F(x))}{\Psi(x)} \, dx \leq K \int_0^a \frac{\Phi(xf(x))}{\Psi(x)} \, dx.
\]

**Proof.** Integration by parts and the given hypothesis yield
\[
\int_0^a \frac{\Phi(F(x))}{\Psi(x)} \, dx = \frac{a \Phi(F(a))}{\Psi(a)} - \int_0^a \frac{x \phi(F(x)) f(x)}{\Psi(x)} \, dx
\]
\[
\geq - \int_0^a \frac{xf(x) \phi(F(x))}{\Psi(x)} \, dx.
\]

Now using Chen’s inequality ([1], 3.10 for \( t = 1 \)), we get on using \( \Phi(u)/u \leq \phi(u) < \Phi(2u)/u \) and \( \Phi(au) \leq c \Phi(u) \),
\[
x f(x) \phi(F(x)) = \max\{xf(x) \phi(xf(x)), F(x) \phi(F(x))\}
\]
\[
\leq \{ \Phi(2xf(x)) + \Phi(2F(x)) \} \leq c[\Phi(xf(x)) + \Phi(F(x))].
\]
Therefore
\[ \int_0^a \frac{\Phi(F(x))}{\Psi(x)} \, dx \geq -c \int_0^a \frac{\Phi(xf(x))}{\Psi(x)} \, dx - c \int_0^a \frac{\Phi(F(x))}{\Psi(x)} \, dx \]
whence
\[ \int_0^a \frac{\Phi(F(x))}{\Psi(x)} \, dx \leq K \int_0^a \frac{\Phi(xf(x))}{\Psi(x)} \, dx. \]

4. Proof of the Theorem. We shall prove the following implications: (2.1) \iff (2.4); (2.4) \implies (2.2); (2.2) \implies (2.3) and (2.3) \implies (2.4).

(2.1) \implies (2.4). In fact, if we write \( 1 - x = y \), then we have for \( 1/(n + 1) < y \leq 1/n \), \( n \geq 2 \),

\[ f(1 - y) \geq \sum_{k=0}^n a_k (1 - y)^k \geq (1 - 1/n)^n \sum_{k=0}^n a_k \geq \frac{A_n^4}{4}. \]

Thus
\[ f(1 - y) \geq KA_n \quad \text{for} \quad 1/(n + 1) < y \leq 1/n \quad (n = 2, 3, \ldots ). \]

Now
\[
\sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2 \Psi(1/n)} \leq K \sum_{n=1}^{\infty} \int_n^{n+1} \frac{t^2 \Phi(A_{[t]})}{\Psi(1/t)} \, dt \\
= K \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} \frac{\Phi(A_{[1/u]})}{\Psi(u)} \, du \\
= K \int_{1/2}^1 \frac{\Phi(A_{[1/u]})}{\Psi(u)} \, du + K \sum_{n=2}^{\infty} \int_{1/(n+1)}^{1/n} \frac{\Phi(A_{[1/u]})}{\Psi(u)} \, du \\
\leq K + K \int_0^{1/2} \frac{\Phi(f(x))}{\Psi(1 - x)} \, dx < \infty.
\]

(2.4) \implies (2.1). We have
\[
\int_0^1 \frac{\Phi(f(x))}{\Psi(1 - x)} \, dx = \sum_{n=2}^{\infty} \int_{1/(n-1)}^{1/n} \frac{\Phi(f(x))}{\Psi(1 - x)} \, dx \\
= \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} \Phi(1 - x) \, dx \\
= \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} \Phi(1 - x) \Psi(x) \, dx \\
= \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} \frac{\Phi(1 - x)}{\Psi(x)} \, dx.
\]
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\[ \left| \text{Re} f(e^{ix}) \right| = \left| \sum_{k=1}^{\infty} a_k \cos kx \right| \\
= \left| \sum_{k=1}^{n} a_k \cos kx + \sum_{k=n+1}^{\infty} a_k \cos kx \right| \\
= \left| \sum_{k=1}^{n} a_k \cos kx + \sum_{k=n}^{\infty} (a_k - a_{k+1}) D_k(x) - a_n D_n(x) \right| \\
\leq A_n + O(1/x) \sum_{k=n}^{\infty} |a_k - a_{k+1}| + O(x^{-1} a_n). \\

Then, on account of (2.0), it follows that

\[ \int_{0}^{\pi} \frac{\Phi\left(\left|\text{Re} f(e^{ix})\right|\right)}{\Psi(x)} \, dx = \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \frac{\Phi\left(\left|\text{Re} f(e^{ix})\right|\right)}{\Psi(x)} \, dx \]

\[ = \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \Phi\left[A_n + O(1/x)(a_n) + O(1/x)(a_n)\right] \, dx \]

\[ \leq K \sum_{n=1}^{\infty} \frac{\Phi(A_n + na_n)}{n^2 \Psi(1/n)}. \]

Since \( \langle n^{-\beta} a_n \rangle \) is monotonically decreasing,

\[ A_n = \sum_{k=1}^{n} k^{-\beta} a_k k^\beta \geq n^{-\beta} a_n \sum_{k=1}^{n} k^\beta \geq K n^{-\beta} a_n n^{\beta+1} = Kn a_n. \]

Therefore, we have

\[ \int_{0}^{\pi} \frac{\Phi\left(\left|\text{Re} f(e^{ix})\right|\right)}{\Psi(x)} \, dx \leq K \sum_{n=1}^{\infty} \frac{\Phi(A_n)}{n^2 \Psi(1/n)} < \infty. \]
It can be proved similarly that

\[
\int_0^\pi \frac{\Phi\left(\left|\Im f(e^{i\theta})\right|/\Psi(x)\right)}{\Psi(x)}\,dx < \infty.
\]

(2.2) \Rightarrow (2.3). Let \( r(t) = \text{Re } f(e^{it}) \). The condition (2.0) implies that \( \langle a_n \rangle \) is a sequence of bounded variation. Hence the Fourier series of \( r \) converges for \( x > 0 \). We write

\[
R(t) = \int_0^t r(x)\,dx \quad \text{and} \quad R_1(t) = \int_0^t R(x)\,dx.
\]

Integrating the Fourier series of \( r(t) \), we have

\[
R_1(t) = \sum_{j=1}^{\infty} a_j (1 - \cos jt) j^{-2}
\]

\[
\geq \sum_{j=1}^{n} a_j j^{-2} (1 - \cos jt)
\]

\[
= 2 \sum_{j=1}^{n} j^{-2} a_j \frac{\sin^2 j t}{2}
\]

\[
\geq 2 \sum_{j=1}^{n} j^{-2} a_j \frac{4 j^2 t^2}{\pi^2}, \quad \pi/(n + 1) \leq t \leq \pi/n
\]

\[
\geq K t^2 \sum_{j=1}^{n} a_j \geq K t^2 a_n \quad \text{due to the monotonicity of} \quad \langle n^{-\beta} a_n \rangle.
\]

Now, by Lemma 1, we have

\[
\sum_{n=1}^{\infty} \frac{\Phi(n a_n)}{n^2 \Psi(1/n)} \leq \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \frac{\Phi(x^{-2} R_1(x))}{\Psi(x)}\,dx
\]

\[
= \int_0^\pi \frac{\Phi\left(x^{-2} R_1(x)\right)}{\Psi(x)}\,dx
\]

\[
\leq \int_0^\pi \frac{\Phi\left(x^{-2} \int_0^x |R(t)|\,dt\right)}{\Psi(x)}\,dx
\]

\[
\leq \int_0^\pi \frac{\Phi\left(x^{-1} \int_0^x \frac{|R(t)|}{t}\,dt\right)}{\Psi(x)}\,dx
\]

\[
\leq K \int_0^\pi \frac{\Phi\left(\frac{|R(x)|}{x}\right)}{\Psi(x)}\,dx
\]

\[
\leq K \int_0^\pi \frac{\Phi\left(|f(e^{i\theta})|\right)}{\Psi(x)}\,dx
\]

\[
\leq K \int_0^\pi \frac{\Phi\left(|f(e^{i\theta})|\right)}{\Psi(x)}\,dx < \infty.
\]
(2.3) ⇒ (2.4). Let \( a(x) \) be a function equal to \( a_n \) if \( n - 1 \leq x < n, \ n = 1, 2, \ldots, \) and let \( A(x) = \int_0^x a(t) \, dt. \) The assumption

\[
\sum_{n=1}^{\infty} \frac{\Phi(na_n)}{n^2\Psi(1/n)} < \infty
\]

implies that \( \Phi(ta(t))/\Psi(t) \) is integrable on the positive half line. Then, by virtue of Lemma 2, \( \int_0^x (\Phi(A(t))/\Psi(t)) \, dt < \infty. \) But this is equivalent to the convergence of the series \( \sum_{n=1}^{\infty} \Phi(A_n)/n^2\Psi(1/n). \) This completes the proof of the theorem.

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REFERENCES

1. Y. M. Chen, Some asymptotic properties of Fourier constants and integrability theorems, Math. Z. 68 (1957), 227–244.
2. ———, Some further asymptotic properties of Fourier constants, Math. Z. 69 (1958), 105–120.

Department of Mathematics, M. D. University, Rohtak - 124001, India

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