PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 2, February 1985

A UNIQUENESS CONDITION FOR SEQUENCES

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ABSTRACT. Under certain conditions a sequence of real numbers converging to zero is determined up to a constant multiple by the comparisons of its subsums. One such condition is that each number in the sequence be majorized by the sum of the elements beyond it.

A uniqueness condition for sequences. A sequence $\langle a_n \rangle$ of real numbers induces an ordering on subsets of N in the following way:

$$I \leqslant J \quad \text{iff} \quad \sum_{i \in I} a_i \leqslant \sum_{j \in J} a_j.$$

We ask the question: Under what conditions does the induced ordering determine the sequence (up to a constant multiple)? A partial answer to this question is given by the theorem below, as well as by the remarks and counterexamples at the end of the paper.

I would like to thank the referee for his extremely helpful suggestions. He also points out that Proposition 1 below is essentially an old Putnam problem (1955A3).

DEFINITIONS. We say that $\langle a_n \rangle$ and $\langle b_n \rangle$ are two *sympathetic* sequences, and write $\langle a_n \rangle \sim \langle b_n \rangle$ iff $\langle a_n \rangle$ and $\langle b_n \rangle$ induce the same ordering on subsets of N. We note that if $\langle a_n \rangle \sim \langle b_n \rangle$, then $\sum_{i \in I} a_i = \sum_{j \in J} a_j$ iff $\sum_{i \in I} b_i = \sum_{j \in J} b_j$. Also, if $a_n > \sum_{k=n+1}^{\infty} a_k$ we say that a_n is a *bully*. The existence of bullies in a sequence is strongly related to the original question. In particular we have the following.

THEOREM. *Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two nonincreasing sequences of real numbers with no bullies such that $a_n > 0$, $b_n > 0$, $a_n \to 0$, $b_n \to 0$ and $\langle a_n \rangle \sim \langle b_n \rangle$. Then there is an α such that $a_n = \alpha b_n$ for all n.*

COROLLARY. *If, in addition, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = 1$, then $a_n = b_n$ for all n.*

The corollary answers a question of Chauqui and Malitz. It states that a probability measure on a σ-complete atomic Boolean algebra is unique if it respects a preordering (a total, transitive, complete extension of the Boolean ordering) which includes the condition $a_n \leqslant \sum_{k=n+1}^{\infty} a_k$ for all n (no bullies in the Boolean ordering). Chuaqui and Malitz establish the existence of a compatible measure in this case assuming certain natural conditions on the preordering [CM]. Their conditions are a (nearly complete) characterization of which orderings on 2^N are induced by sequences.

Received by the editors May 31, 1983 and, in revised form, February 8, 1984.
1980 *Mathematics Subject Classification*. Primary 40B05, 40A10, 26D15.

PROPOSITION 1. *Suppose $\langle u_n \rangle$ is a nonincreasing sequence with $u_n \to 0$ and no bullies and let v be any real number such that $0 \leqslant v \leqslant \Sigma_{k=1}^{\infty} u_k$. Then there exists an infinite subset $I \subseteq N$ such that $v = \Sigma_I u_i$. (Note. If Σu_n diverges, then there are no bullies and v may be any nonnegative number.)*

PROOF. Define I inductively by the Greedy Algorithm. If $v > u$, then $1 \in I$; once $I_n (= I \cap \{1, \dots, n\})$ has been determined, $n + 1 \in I_{n+1}$ iff

$$(*) \qquad \sum_{I_n} u_i = i_{n+1} < v.$$

Define $v_n = \Sigma_{I_n} u_i$. We note that $v > v_n \geqslant v_{n-1}$, so that $v_n \uparrow v_0 \leqslant v$. If $v_0 = v$, we are done. Otherwise, let $d = v - v_0 > 0$. From $(*)$, if $r \notin I$, then $v_{r-1} + u_r > v$ so that $u_r > d$. Thus only finitely many r are not in I. If no indices are omitted, then $I = N$ and $\Sigma_{k=1}^{\infty} u_k < v$, a contradiction. Otherwise let s be the last omitted index. Then

$$v_{s-1} + u_s \geqslant x > v_{s-1} + \sum_{k=s+1}^{\infty} u_k,$$

so that u_s is a bully, a contradiction.

PROPOSITION 2. *For j fixed, let $M_j = \{i\colon b_i/a_i \neq b_j/a_j\}$. If $\langle a_n \rangle \sim \langle b_n \rangle$, then $\Sigma_{M_j} a_k$ and $\Sigma_{M_j} b_k$ are both finite.*

COROLLARY. *If $\langle a_n \rangle \sim \langle b_n \rangle$ and $\langle b_n \rangle$ is not a multiple of $\langle a_n \rangle$, then $\Sigma a_i < \infty$ and $\Sigma b_i < \infty$.*

PROOF. Suppose $b_r \neq b_1 a_r / a_1$ for some r. Since $N = M_1 \cup M_r$, $\Sigma a_i \leqslant \Sigma_{M_1} a_i + \Sigma_{M_r} a_i < \infty$. Similarly, $\Sigma b_i < \infty$.

PROOF OF PROPOSITION 2. Let $k_r = b_r/a_r$, $M_j^+ = \{i\colon k_i > k_j\}$ and $M_j^- = \{i\colon k_i < k_j\}$. Since $M_j = M_j^+ \cup M_j^-$ and the situation is symmetric, it suffices to prove that $\Sigma_{M_j^+} a_i$ and $\Sigma_{M_j^+} b_i$ are both finite. Suppose not; then $b_i > k_j a_i$ on M_j^+ implies that $\Sigma_{M_j^+} b_i$ diverges if either diverges. Choose $s > j$ so that $s \in M_j^+$, $a_j > a_s$ and $b_j > b_s$ and write $b_s = k_j a_s + \varepsilon$. Let $x = b_j - b_s + \varepsilon/2$. We assume (for contradiction) that $\Sigma_{M_j^+} b_i$ diverges, so that Proposition 1 can now be applied to $\langle b_n \rangle$ restricted to $M_j^+ - \{s\}$. Choose $J_0 \subset M_j^+ - \{s\}$ such that $\Sigma_{J_0} b_i = x$ and let $J = J_0 \subset \{s\}$. Then $\Sigma_J b_i = b_s + x = b_j + \varepsilon/2 > b_j$. On the other hand,

$$\begin{aligned}\sum_J a_i &= a_s + \sum_{J_0} a_i \leqslant a_s + k_j^{-1} \sum_{J_0} b_i = a_s + k_j^{-1}(b_j - b_r + \varepsilon/2) \\ &= k_j^{-1}(b_r - \varepsilon + b_j - b_r + \varepsilon/2) = k_j^{-1}(b_j - \varepsilon/2) < a_j.\end{aligned}$$

This violates the sympathy of $\langle a_n \rangle$ and $\langle b_n \rangle$.

LEMMA. *If $\langle a_n \rangle \sim \langle b_n \rangle$ and $\Sigma a_n = \Sigma b_n$, then there are infinitely many n such that $a_n \leqslant b_n$, and infinitely many n such that $a_n \geqslant b_n$.*

PROOF. Suppose not, and obtain an N such that (w.l.o.g.), for all $k > N$, $a_k > b_k$. By Proposition 1 we may write $a_N = \Sigma_I a_i$, where I is infinite (so that $\min I > N$), and since $\langle a_n \rangle \sim \langle b_n \rangle$ $b_N = \Sigma_I b_i$, so that $a_N > b_N$. Thus (by backward induction), $a_n > b_n$ for all n, and so $\Sigma a_n > \Sigma b_n$, a contradiction.

Proof of the Theorem. By the Corollary to Proposition 2 it suffices to prove the corollary to the Theorem. Define

$$f(x)=\sum_{I_x} b_i, \quad \text{where } I_x \text{ is such that } x=\sum_{I_x} a_i.$$

Note that such an I_x always exists for any $x \in [0,1]$ by Proposition 1. Since $\langle a\rangle \sim \langle b\rangle$, it is clear that f is well defined and that if $x \leqslant y$, then $f(x) \leqslant f(y)$. Thus f is a monotone function taking $[0,1]$ into $[0,1]$, with $f(0)=0$ and $f(1)=1$.

By Lebesgue's theorem on monotone functions, this implies that $f'(x)$ exists a.e., and that $\int_a^b f'(x)\,dx \leqslant f(b)-f(a)$ for any $[a,b] \subseteq [0,1]$ (see, for example, [**R**, p. 96]). Now pick any x such that $f'(x)$ exists, and obtain I_x as before. Define

$$h_n=\begin{cases} a_n & \text{if } n \notin I_x, \\ -a_n & \text{if } n \in I_x. \end{cases}$$

Then if $a_n \notin I_x$,

$$\frac{f(x+h_n)-f(x)}{h_n}=\frac{\Sigma_{I_x} b_i + b_n - \Sigma_{I_x} b_i}{a_n}=\frac{b_n}{a_n},$$

while $a_n \in I_x$ implies

$$\frac{f(x+h_n)-f(x)}{h_n}=\frac{\Sigma_{I_x} b_i - b_n - \Sigma_{I_x} b_i}{-a_n}=\frac{b_n}{a_n}.$$

We now have $\lim_{n\to\infty}(b_n/a_n)$ exists, and so by the Lemma $\lim_{n\to\infty}(b_n/a_n)=1$, and thus $f'(x)=1$ a.e.

Finally, suppose that there is a point c such that $f(c)\neq c$. If $f(c)>c$, then

$$\int_c^1 f'(x)\,dx = 1-c > f(1)-f(c),$$

while $f(c)<c$ yields

$$\int_0^c f'(x)\,dx = c-0 > f(c)-f(0).$$

Since either situation is impossible, it must be that $f(x)=x$ on $[0,1]$, and that, in particular, $a_n=b_n$ for all n.

Remark. It is evident that the condition "no bullies" is necessary in order to have every $x \in [0,\Sigma a_n]$ expressible as a sum of a_i's, for if $a_n > \Sigma_{k=n+1}^{\infty} a_k$, then any x between the two cannot possibly be so written.

Necessary and sufficient conditions for the conclusion of the Theorem are more elusive. The following examples will establish: (1) The Theorem is not true if the condition "no bullies" is removed. (2) It is possible that the ordering induced by $\langle a_n\rangle$ determines the sequence up to a constant multiple but that $\langle a_n\rangle$ has no bullies.

Example 1. Any two sequences $\langle a_n\rangle$ and $\langle b_n\rangle$ in which every element is a bully are clearly sympathetic. Thus, for example, the sequences $\langle ar^n\rangle$ with $0<r<\frac{1}{2}$ are all sympathetic. More generally, call two finite sequences $\{d_1,\ldots,d_r\}$ and $\{e_1,\ldots,e_r\}$ r-sympathetic if, for $I, J \subset \{1,\ldots,r\}$, $\Sigma_I d_i \geqslant \Sigma_J d_j$ iff $\Sigma_I e_i \geqslant \Sigma_J e_j$. It is evident that there are many sets of r-sympathetic sequences which are not proportional. Now construct sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ out of blocks of $r(i)$-sympathetic strings, $r(i) \geqslant 1$:

$$\langle a \rangle = \left\{ d_1^1, \ldots, d_{r(1)}^1, d^2, \ldots, d_{r(2)}^2, \ldots \right\},$$
$$\langle b \rangle = \left\{ e^1, \ldots, e_{r(1)}^1, e^2, \ldots, e_{r(2)}^2, \ldots \right\}$$

subject to the conditions

$$\sum_I d_i^k - \sum_J d_j^k > \sum_{n=k+1}^{\infty} \sum_{m=1}^{r(n)} d_m^n \quad \text{and} \quad \sum_I e_i^k - \sum_J e_j^k > \sum_{n=k+1}^{\infty} \sum_{m=1}^{r(n)} e_m^n$$

whenever the left sides are *positive* and $I, J \subset \{1, \ldots, r(k)\}$.

Now, the ordering on 2^N induced by $\langle a_n \rangle$ is given by:

$I = J$ iff the order induced within each block is equal.

$I > J$ iff the first block in which the induced order is unequal gives $I_k > J_k$, where I_k and J_k are the elements of I and J within the kth block. By construction and the block-by-block sympathy of $\langle a_n \rangle$ and $\langle b_n \rangle$ it is clear that $\langle b_n \rangle$ induces the same order so that $\langle a_n \rangle \sim \langle b_n \rangle$, although they need not be proportional.

EXAMPLE 2. Let $\langle a_n \rangle$ be the sequence whose sum is one determined by the following set of equalities:

$$a_{3n+1} = a_{3n+2} + \sum_{k=3n+4}^{\infty} a_k, \quad a_{3n+2} = \sum_{k=3n+3}^{\infty} a_k,$$
$$a_{3n+1} + \sum_{k=3n+4}^{\infty} a_k = a_{3n+2} + a_{3n+3}.$$

The inequalities for $n = 0$ force $a_1 = 4/10$, $a_2 = 3/10$, $a_3 = 2/10$ and $\Sigma_{k=4}^{\infty} a_k = 1/10$ (so that $a_3 > \Sigma_{k=4}^{\infty} a_k$). This can easily be seen by substituting $4/10 + \varepsilon_1$, $3/10 + \varepsilon_2$, $2/10 + \varepsilon_3$ and $1/10 + \varepsilon_4$ for a_1, a_2, a_3 and $\Sigma_{k=4} a_k$, respectively, and checking that the only solution (given that $\Sigma a_k = 1$) is $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$. In exactly the same way it can be checked that $a_{3n+1} = 4/10^{n+1}$, $a_{3n+2} = 3/10^{n+1}$, $a_{3n+3} = 2/10^{n+1}$ and $\Sigma_{k=3n+4}^{\infty} a_k = 1/10^{n+1}$, so that every a_{3n+3} is a bully, even though the induced order determines $\langle a_n \rangle$ uniquely.

Open questions. (1) What are necessary and sufficient conditions for the conclusion of the theorem? An answer to this question might be of use in anwering open questions in **[CM]** also.

In the absence of a complete answer to (1), the following might be interesting:

(2) Are all sympathetic sequences which are not proportional given by block bullies as in Example 1?

(3) In Example 2 we had uniqueness even though every third element was a bully. Could we have every other element a bully?

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