A RIGIDITY RESULT FOR HOLOMORPHIC IMMERSIONS OF SURFACES IN CPⁿ

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ABSTRACT. A pinching condition for the Gaussian curvature implies rigidity.

In this paper we prove a stronger version of a rigidity result obtained by Lawson [3] (the quantization lemma). Toward this aim we briefly recall some results and formulas for holomorphic immersions of a Riemann surface into projective space; we refer the reader to [1–3] for details.

Let $R$ be a Riemann surface and $f : R \to \mathbb{C}P^n$ a holomorphic immersion, where $\mathbb{C}P^n$ is the complex $n$-dimensional projective space with the normalized Fubini-Study metric of constant holomorphic sectional curvature 1.

Let $z$ be a local complex coordinate on $R$. We can express the metric induced by $f$ on $R$ as

\[ ds^2 = 2Fdz \, d\bar{z}, \]

where $F$ is a positive smooth function. Then the Laplace operator and the Gauss curvature $K$ of this metric are respectively given by

\[ \Delta = \frac{2}{F} \frac{d}{dz} \frac{d}{d\bar{z}}, \]

\[ K = \frac{1}{F} \frac{d}{dz} \frac{d}{d\bar{z}} \log F. \]

If the image $f(R)$ in $\mathbb{C}P^n$ lies in no proper, totally geodesic submanifold of $\mathbb{C}P^n$, then Calabi [2] showed that it is possible to define inductively a sequence of functions $\{F_k\}$, $k = 0, \ldots, n+1$, by setting

\[ F_0 = 1, \quad F_1 = F, \]

\[ F_{k+1} = \frac{F_k^2}{F_{k-1}} \left( \frac{d}{dz} \frac{d}{d\bar{z}} \log F_k + \frac{k+1}{2} F \right) \]

for $k = 1, \ldots, n$ and they satisfy the following:

(5.i) For $0 \leq k \leq n$, $F_k$ is nonnegative and vanishes only at isolated points.

(5.ii) The succeeding function $F_{k+1}$ is defined by (4) away from those points but extends to a real analytic function on all of $R$.

(5.iii) $F_{n+1} \equiv 0$.

Using the sequence $\{F_k\}$, the curvature functions are inductively defined by

\[ K_k = \frac{F_{k+1}F_{k-1}}{FF_k^2} \quad \text{for} \quad 1 \leq k \leq n. \]
We remark that the curvature functions are independent of local coordinates and that $K_n \equiv 0$. Moreover $K_k \geq 0$ and, for $1 \leq k < n$, the set of zeros $Z_{K_k}$ contains only isolated points.

From the recursive definition (5) of the curvature functions it is not hard to see that the following nontrivial relations hold:

\begin{equation}
K_1 = 1 - K, \tag{6}
\end{equation}

\begin{equation}
\frac{1}{2} \Delta \log K_k = K_{k+1} + K_{k-1} - 2K_k + K \quad \text{for } k = 1, \ldots, n-1, \tag{7}
\end{equation}

where $K_0 = \frac{1}{2}$.

By induction from (6) and (7) we obtain

\begin{equation}
\Delta \log K_1, K_2, \ldots, K_n = n(n+1)(K - 1/n). \tag{8}
\end{equation}

Of course formulas (7) and (8) are valid on their appropriate domains of definition.

Now suppose $R$ to be compact. Then the integrated $k$th curvature $\nu_k$ is defined to be

\begin{equation}
\nu_k = \frac{1}{2\pi} \int_R K_k \, dA, \quad k = 0, \ldots, n, \tag{9}
\end{equation}

where $dA$ is the area element of the Riemann surface $R$. It is well known that $\nu_k$ is an integer, precisely the order of the $k$th osculating curve of the isometric immersion of $R$ into $\mathbb{C}P^n$. In particular, $4\pi \nu_0 = A$, the area of $f: R \to \mathbb{C}P^n$.

Define $\sigma_k$ to be

\begin{equation}
\sigma_k = -\frac{1}{4\pi} \int_R \Delta \log K_k \, dA, \quad k = 0, \ldots, n. \tag{10}
\end{equation}

The $\sigma_k$'s and the $\nu_k$'s are related by the classical Plücker formulas

\begin{equation}
\sigma_k = 2\nu_k - \nu_{k+1} - \nu_{k-1} + 2(g-1), \quad k = 0, \ldots, n-1. \tag{11}
\end{equation}

where $g$ is the genus of $R$ and $\nu_{-1} = 0$. With $C_n$ we will indicate Calabi's curve (see [1]) in $\mathbb{C}P^n$ of constant Gaussian curvature $1/n$. $C_n$ does not lie in any linear subspace of $\mathbb{C}P^n$ and is given by the following imbedding of $\mathbb{C}P^1$ in $\mathbb{C}P^n$:

\begin{equation}
(z_0, z_1) \rightarrow \left( z_0^n, \sqrt{n}z_0^{n-1}z_1, \ldots, \sqrt{\binom{n}{k}}z_0^{n-k}z_1^k, \ldots, z_1^n \right). \tag{12}
\end{equation}

The curvature functions for this curve are

\begin{equation}
K_k = \frac{(k+1)}{2n}(n-k). \tag{13}
\end{equation}

Calabi [1] showed that, modulo holomorphic congruences, $C_n$ is the only curve in $\mathbb{C}P^n$ of constant Gaussian curvature which does not lie in any linear subspace.

We are now ready to state and prove the following result.

**Theorem.** Let $R$ be a compact, connected Riemann surface (without boundary) and let $f: R \to \mathbb{C}P^n$ be a holomorphic immersion such that $f(R)$ does not lie in any linear subspace of $\mathbb{C}P^n$. Let $K$ be the Gaussian curvature of the induced metric. If

\begin{equation}
K \geq 1/n, \tag{14}
\end{equation}

then $f$ is an isometric immersion into $\mathbb{C}P^n$. In particular, $\nu_k$ is an integer for $k = 0, \ldots, n$. The Plücker formulas (11) hold and $\sigma_k = \nu_k$ for $k = 0, \ldots, n$.

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then $R$ is topologically a sphere, $K \equiv 1/n$ and $f(R) = C_n$, where $C_n$ is defined above.

**Proof.** First of all we observe that $Z_{K_1} \cup \cdots \cup Z_{K_{n-1}} \neq R$. From (8) and (14) we obtain

$$\Delta \log K_1^{n-1}K_2^{n-2}, \ldots, K_{n-1} \geq 0 \text{ on } R \setminus (Z_{K_1} \cup \cdots \cup Z_{K_{n-1}}).$$

Now let $p$ be a point of $R$ in which $K_1^{n-1} \cdots K_{n-1}$ attains its positive absolute maximum. Then (15) is valid in a neighborhood $U$ of $p$ and at $p \log K_1^{n-1} \cdots K_{n-1}$ attains its absolute maximum. The maximum principle for subharmonic functions (see [5]) then applies to show that $\log K_1^{n-1} \cdots K_{n-1}$, and hence $K_1^{n-1} \cdots K_{n-1}$, is constant on $U$. A standard argument using connectedness of $R$ then shows that

$$K_1^{n-1} \cdots K_{n-1} = \text{constant on } R.$$

From (8) and (16) we deduce $K \equiv 1/n$. Then the Gauss-Bonnet theorem applies to show $g = 0$, completing the proof.

Suppose $R$ to be compact. Then integrating (8) we obtain the following estimate for the area of the immersion:

$$A = 4\pi n(1 - g) + \frac{4\pi}{n+1} \sum_{k=1}^{n-1} (n-k)\sigma_k.$$

In particular, for a holomorphic immersion of a sphere such that $f(R)$ does not lie in any subspace of $\mathbb{C}P^n$ we have

$$A \geq 4\pi n.$$

Remark that if $Z_{K_k} = \emptyset$, then $\sigma_k = 0$ because of (10); actually the converse is true because of the following

**Lemma.** Let $R$ be a compact Riemann surface, $h: R \to R$ be a smooth positive function whose zeros set $Z$ contains only isolated points and

$$\Delta \log h = f \text{ on } R \setminus Z$$

for some continuous $f: R \to R$. Then

$$\int_R \Delta \log h \leq 0$$

and equality holds if and only if $Z = \emptyset$.

We omit the computational proof of the lemma.

We have the following

**Proposition.** In the same hypothesis of the theorem, the following are equivalent:

(i) $\nu_0 \leq n(1 - g)$,
(ii) $Z_{K_1} = \cdots = Z_{K_{n-1}} = \emptyset$,
(ii) $g = 0, \nu_k = (k+1)(n-k), k = 0, \ldots, n-1$.

**Proof.** (i)$\to$(ii). Indeed, since $A = 4\pi \nu_0$ from (i) and (17), we get $\sigma_k = 0$, $k = 1, \ldots, n-1$; hence, by the above remark, (ii).

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(ii)→(iii). From (ii) and the above remark, we have $\sigma_k = 0$, $k = 1, \ldots, n - 1$; hence from (11) we obtain the system of equations
\[
0 = 2v_k - v_{k+1} - v_{k-1} + 2(g - 1), \quad k = 0, \ldots, n - 1.
\]
It is easy to see that the solutions of (21) are given by
\[
\nu_k = (k + 1)(v_0 + k(g - 1)), \quad k = 0, \ldots, n,
\]
and since $\nu_n = 0$ we obtain $\nu_0 = n(1 - g)$. On the other hand $A = 4\pi \nu_0$ and we deduce, using the Gauss-Bonnet theorem, that $g = 0$ and hence (iii).

(iii)→(i). It is obvious.

REMARKS. (1) In the case $n = 2$ from (6) we immediately obtain from $K \leq \frac{1}{2}$ that (ii) holds, hence (i) and using the Gauss-Bonnet theorem we get $K \equiv \frac{1}{2}$. This case was considered by Nomizu and Smyth [4]. The previous proposition should be compared with Theorem 3 in Lawson [3].

(2) If we suppose
\[
K_2 \geq 3 \left( \frac{1}{2} - K \right),
\]
then from (7) we have $\Delta \log K_1 \geq 0$ and we deduce as in the theorem that $K_1$, and hence $K$, is constant. Again we observe that in the case $n = 2$ (23) becomes $K \geq \frac{1}{2}$.

REFERENCES