A RIGIDITY RESULT FOR HOLOMORPHIC IMMERSIONS OF SURFACES IN $\mathbf{CP}^n$

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ABSTRACT. A pinching condition for the Gaussian curvature implies rigidity.

In this paper we prove a stronger version of a rigidity result obtained by Lawson [3] (the quantization lemma). Toward this aim we briefly recall some results and formulas for holomorphic immersions of a Riemann surface into projective space; we refer the reader to [1-3] for details.

Let $R$ be a Riemann surface and $f: R \rightarrow \mathbf{CP}^n$ a holomorphic immersion, where $\mathbf{CP}^n$ is the complex $n$-dimensional projective space with the normalized Fubini-Study metric of constant holomorphic sectional curvature 1.

Let $z$ be a local complex coordinate on $R$. We can express the metric induced by $f$ on $R$ as

\begin{equation}
\text{ds}^2 = 2F \text{dz} \text{d\bar{z}},
\end{equation}

where $F$ is a positive smooth function. Then the Laplace operator and the Gauss curvature $K$ of this metric are respectively given by

\begin{align}
\Delta &= \frac{2}{F} \frac{d^2}{dz \, d\bar{z}}, \\
K &= -\frac{1}{F} \frac{d}{dz} \frac{d}{d\bar{z}} \log F.
\end{align}

If the image $f(R)$ in $\mathbf{CP}^n$ lies in no proper, totally geodesic submanifold of $\mathbf{CP}^n$, then Calabi [2] showed that it is possible to define inductively a sequence of functions $\{F_k\}$, $k = 0, \ldots, n+1$, by setting

\begin{align}
F_0 &= 1, \\
F_1 &= F, \\
F_{k+1} &= \frac{F_k^2}{F_{k-1}} \left( \frac{d}{dz} \frac{d}{d\bar{z}} \log F_k + \frac{k+1}{2} F_k \right)
\end{align}

for $k = 1, \ldots, n$ and they satisfy the following:

(5.i) For $0 \leq k \leq n$, $F_k$ is nonnegative and vanishes only at isolated points.
(5.ii) The succeeding function $F_{k+1}$ is defined by (4) away from those points but extends to a real analytic function on all of $R$.
(5.iii) $F_{n+1} \equiv 0$.

Using the sequence $\{F_k\}$, the curvature functions are inductively defined by

\begin{equation}
K_k = \frac{F_{k+1} F_{k-1}}{F_k F_k^2} \quad \text{for} \quad 1 \leq k \leq n.
\end{equation}
We remark that the curvature functions are independent of local coordinates and that $K_n = 0$. Moreover $K_k > 0$ and, for $1 \leq k < n$, the set of zeros $Z_{K_k}$ contains only isolated points.

From the recursive definition (5) of the curvature functions it is not hard to see that the following nontrivial relations hold:

(6) $K_1 = 1 - K$,

(7) $\frac{1}{2} \Delta \log K_k = K_{k+1} + K_{k-1} - 2K_k + K$ for $k = 1, \ldots, n - 1$,

where $K_0 = \frac{1}{2}$.

By induction from (6) and (7) we obtain

(8) $\Delta \log K_{n-1}^n K_{n-2} \ldots, K_{n-1} = n(n + 1)(K - 1/n)$.

Of course formulas (7) and (8) are valid on their appropriate domains of definition.

Now suppose $R$ to be compact. Then the integrated kth curvature $\nu_k$ is defined to be

(9) $\nu_k = \frac{1}{2\pi} \int_R K_k dA$, $k = 0, \ldots, n$,

where $dA$ is the area element of the Riemann surface $R$. It is well known that $\nu_k$ is an integer, precisely the order of the kth osculating curve of the isometric immersion of $R$ into $\mathbb{C}P^n$. In particular, $4\pi \nu_0 = A$, the area of $f: R \rightarrow \mathbb{C}P^n$.

Define $\sigma_k$ to be

(10) $\sigma_k = \frac{1}{4\pi} \int_R \Delta \log K_k dA$, $k = 0, \ldots, n$.

The $\sigma_k$'s and the $\nu_k$'s are related by the classical Plücker formulas

(11) $\sigma_k = 2\nu_k - \nu_{k+1} - \nu_{k-1} + 2(g - 1)$, $k = 0, \ldots, n - 1$.

where $g$ is the genus of $R$ and $\nu_{-1} = 0$. With $C_n$ we will indicate Calabi's curve (see [1]) in $\mathbb{C}P^n$ of constant Gaussian curvature $1/n$. $C_n$ does not lie in any linear subspace of $\mathbb{C}P^n$ and is given by the following imbedding of $\mathbb{C}P^1$ in $\mathbb{C}P^n$:

(12) $(z_0, z_1) \rightarrow \left( z_0^n, \sqrt{n} z_0^{n-1} z_1, \ldots, \sqrt{\binom{n}{k}} z_0^{n-k} z_1^k, \ldots, z_1^n \right)$.

The curvature functions for this curve are

(13) $K_k = \frac{(k + 1)}{2n}(n - k)$.

Calabi [1] showed that, modulo holomorphic congruences, $C_n$ is the only curve in $\mathbb{C}P^n$ of constant Gaussian curvature which does not lie in any linear subspace.

We are now ready to state and prove the following result.

THEOREM. Let $R$ be a compact, connected Riemann surface (without boundary) and let $f: R \rightarrow \mathbb{C}P^n$ be a holomorphic immersion such that $f(R)$ does not lie in any linear subspace of $\mathbb{C}P^n$. Let $K$ be the Gaussian curvature of the induced metric. If

(14) $K \geq 1/n$,
then \( R \) is topologically a sphere, \( K \equiv 1/n \) and \( f(R) = C_n \), where \( C_n \) is defined above.

**Proof.** First of all we observe that \( Z_{K_1} \cup \cdots \cup Z_{K_{n-1}} \neq R \). From (8) and (14) we obtain

\[
(15) \quad \Delta \log K_1^{n-1} K_{k-2}^{n-2}, \ldots, K_{n-1} \geq 0 \quad \text{on} \quad R \setminus (Z_{K_1} \cup \cdots \cup Z_{K_{n-1}}).
\]

Now let \( p \) be a point of \( R \) in which \( K_1^{n-1} \cdots K_{n-1} \) attains its positive absolute maximum. Then (15) is valid in a neighborhood \( U \) of \( p \) and at \( p \) \( \log K_1^{n-1} \cdots K_{n-1} \) attains its absolute maximum. The maximum principle for subharmonic functions (see [5]) then applies to show that \( \log K_1^{n-1} \cdots K_{n-1} \), and hence \( K_1^{n-1} \cdots K_{n-1} \), is constant on \( U \). A standard argument using connectedness of \( R \) then shows that

\[
(16) \quad K_1^{n-1} \cdots K_{n-1} = \text{constant on} \quad R.
\]

From (8) and (16) we deduce \( K \equiv 1/n \). Then the Gauss-Bonnet theorem applies to show \( g = 0 \), completing the proof.

Suppose \( R \) to be compact. Then integrating (8) we obtain the following estimate for the area of the immersion:

\[
(17) \quad A = 4\pi n(1 - g) + \frac{4\pi}{n+1} \sum_{k=1}^{n-1} (n-k)\sigma_k.
\]

In particular, for a holomorphic immersion of a sphere such that \( f(R) \) does not lie in any subspace of \( CP^n \) we have

\[
(18) \quad A \geq 4\pi n.
\]

Remark that if \( Z_{K_k} = \emptyset \), then \( \sigma_k = 0 \) because of (10); actually the converse is true because of the following

**Lemma.** Let \( R \) be a compact Riemann surface, \( h: R \to R \) be a smooth positive function whose zeros set \( Z \) contains only isolated points and

\[
(19) \quad \Delta \log h = f \quad \text{on} \quad R \setminus Z
\]

for some continuous \( f: R \to R \). Then

\[
(20) \quad \int_R \Delta \log h \leq 0
\]

and equality holds if and only if \( Z = \emptyset \).

We omit the computational proof of the lemma.

We have the following

**Proposition.** In the same hypothesis of the theorem, the following are equivalent:

(i) \( \nu_0 \leq n(1 - g) \),

(ii) \( Z_{K_1} = \cdots = Z_{K_{n-1}} = \emptyset \),

(iii) \( g = 0, \nu_k = (k+1)(n-k), \quad k = 0, \ldots, n-1 \).

**Proof.** (i)\(\Rightarrow\)(ii). Indeed, since \( A = 4\pi \nu_0 \) from (i) and (17), we get \( \sigma_k = 0 \), \( k = 1, \ldots, n-1 \); hence, by the above remark, (ii).

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(ii)→(iii). From (ii) and the above remark, we have $\sigma_k = 0, \ k = 1, \ldots, n-1$; hence from (11) we obtain the system of equations

\begin{equation}
0 = 2\nu_k - \nu_{k+1} - \nu_{k-1} + 2(g-1), \quad k = 0, \ldots, n-1.
\end{equation}

It is easy to see that the solutions of (21) are given by

\begin{equation}
\nu_k = (k+1)(\nu_0 + k(g-1)), \quad k = 0, \ldots, n,
\end{equation}

and since $\nu_n = 0$ we obtain $\nu_0 = n(1-g)$. On the other hand $A = 4\pi\nu_0$ and we deduce, using the Gauss-Bonnet theorem, that $g = 0$ and hence (iii).

(iii)→(i). It is obvious.

REMARKS. (1) In the case $n = 2$ from (6) we immediately obtain from $K \leq \frac{1}{2}$ that (ii) holds, hence (i) and using the Gauss-Bonnet theorem we get $K \equiv \frac{1}{2}$. This case was considered by Nomizu and Smyth [4]. The previous proposition should be compared with Theorem 3 in Lawson [3].

(2) If we suppose

\begin{equation}
K_2 \geq 3\left(\frac{1}{2} - K\right),
\end{equation}

then from (7) we have $\Delta \log K_1 \geq 0$ and we deduce as in the theorem that $K_1$, and hence $K$, is constant. Again we observe that in the case $n = 2$ (23) becomes $K \geq \frac{1}{2}$.

REFERENCES


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