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A THEOREM ON THE TENSION FIELD

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ABSTRACT. Suppose M and N are complete Riemannian manifolds; M with Ricci curvature bounded from below and N with sectional curvature bounded from above by a constant K_0. Let $f\colon M \to N$ be a smooth map such that $f(M) \subset B_R$, where B_R is a normal ball in N and furthermore $R < \pi/2\sqrt{K_0}$ if $K_0 > 0$. If the energy density $e(f)$ is bounded below by a positive constant, then there is a point $P \in M$ such that the tension field $\tau(f)$ at P is different from zero.

1. Introduction. Let M be a complete Riemannian manifold with Ricci curvature bounded below by a constant and N be a complete Riemannian manifold with sectional curvature K bounded above by a constant K_0. Suppose $f\colon M \to N$ is a smooth map such that $f(M)$ lies in a normal ball B_R of radius R. From now on, we will moreover suppose that $R < \pi/2\sqrt{K_0}$ if $K_0 > 0$. It is well known that there can be harmonic maps $f\colon M \to N$ with $f(M) \subset B_R$. For example if M is the hyperbolic half plane and N is the euclidean space E^2 then the map $f(z) = (z-i)/(z+i)$, $z \in C$, is harmonic and $f(M)$ is contained in the unit disk D^2 of E^2. We remark that there are points $(z \to \infty)$ on M where the energy density $e(f)$ of f tends to zero. This and all the examples known to us suggested considering whether for any harmonic map $f\colon M \to N$ there must be sequences of points on M where $e(f)$ tends to zero. The answer is given in the following theorem.

THEOREM. *Let $f\colon M \to N$ be a smooth map such that $f(M) \subset B_R$ ($R < \pi/2\sqrt{K_0}$, if $K_0 > 0$). If $e(f)$ is bounded below away from zero then there exists a point $P \in M$ such that the tension field $\tau(f)$ at P is different from zero.*

2. Preliminaries. We denote $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$ the metrics of M and N respectively. Given a smooth map $f\colon M \to N$ the inverse image $f^*(\langle\ ,\ \rangle_N)$ considered as a 2-form on M is positive semidefinite and is called the first fundamental form of f. The induced vector bundle $E = f^{-1}TN \to M$ has a Riemannian structure $\langle\ ,\ \rangle_E$ induced from that of N. Thus if ∇ and $\overline{\nabla}$ are the Riemannian connections in M and N respectively, then for any $Y \in TM$ and for any section $w \in \beta(E)$, where $\beta(E)$ is the set of sections of the bundle E, the induced Riemannian connection $\overline{\overline{\nabla}}$ in E is defined as follows. We consider a local frame field p_a in TN so that for $w \in \beta(E)$ there are real functions h^a in a chart of M with $w = \sum_a h^a p_a$. Then we set

$$\overline{\overline{\nabla}}_Y w = \sum_a \{(Yh^a)p_a + h^a \overline{\nabla}_{f_* Y} p_a\}.$$

We note that the differential df of f, for which we write f_*, can be interpreted as a E-valued 1-form of M i.e., $f_* \in \beta(T^*M \otimes E)$, where T^*M is the dual bundle

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of the tangent bundle TM. The bundle $T^*M \times E \to M$ inherits a Riemannian structure from those of M and E. Thus the norm $\|f_*\|$ of f_* is defined by $\|f_*\|^2 = \sum_i \langle f_*Y_i, f_*Y_i\rangle_N$, where Y_i is an orthonormal basis in M. $\|f_*\|^2$ is known as the energy density of f and it is denoted by $e(f)$. The covariant differential $\tilde{\nabla} f_* \in \beta(T^*M \otimes T^*M \otimes E)$ is called the second fundamental form of f and it is denoted by L. We can see that for vector fields X, Y on M it is valid: $L(X,Y) = \overline{\overline{\nabla}}_X(f_*Y) - f_*(\nabla_X Y)$. The tension field $\tau(f)$ of f is a section of the vector bundle E and it is defined as the trace of the second fundamental form of f. If $\tau(f) \equiv 0$, then f is called harmonic. For the above definitions see [**2**].

We are going to use the following three lemmas.

LEMMA 1. *Let $\Phi\colon M \to R$ be a smooth function having an upper bound. Then for any point $P \in M$ and for any $\varepsilon > 0$, there exists a point $q \in M$ depending on P, such that $\Phi(q) \geq \Phi(P)$, $\|\mathrm{grad}\,\Phi\|(q) < \varepsilon$ and the Laplacian $(\Delta\Phi)(q) < \varepsilon$.*

LEMMA 2. *Suppose x_0 and x are points of N so that x does not lay in the cut locus of x_0. Let $\gamma\colon [0,1] \to N$ be the minimizing geodesic segment connecting x_0 with x parametrized proportionally to the arc length. Take a positive number a. For any Jacobi field X along γ which is zero at $t = 0$ and perpendicular to $\gamma'(t) = T$, we have for $t \in (0,1]$, $\langle \overline{\nabla}_T X, X\rangle_N / \langle X, X\rangle_N \geq \mu(alt)$, where $l = \|T\|_N$ and*

$$\mu(alt) = \begin{cases} al\cot(alt) & \text{if } \max_\gamma K = a^2 \text{ and } l < \pi/a, \\ t^{-1} & \text{if } \max_\gamma K = 0, \\ al\coth(alt) & \text{if } \max_\gamma K = -a^2. \end{cases}$$

From now on, for the sake of economy of notations, we will use the same symbol $\langle\ ,\ \rangle$ instead of $\langle\ ,\ \rangle_M$, $\langle\ ,\ \rangle_N$ and $\langle\ ,\ \rangle_E$.

Let $Q \in N$ be the center of the normal ball B_R. Let Y_P be a unit vector tangent to M at a point $P \in M$ such that $f_*Y_P \neq 0$ and $\gamma(u)$ be a geodesic in M with $\gamma(0) = P$ and $\gamma'(0) = Y_P$. Let $\gamma'(u) = Y$. We join Q with the point $f(\gamma(u))$ by the unique geodesic $\sigma(t,u)$, $0 \leq t \leq 1$, parametrized proportionally to the arc length, such that $\sigma(0,u) = Q$ and $\sigma(1,u) = f(\gamma(u))$. Put $T = \sigma_*\partial/\partial t$, $X = \sigma_*\partial/\partial u$ ($X(0,u) = 0$, $X(1,u) = f_*Y$). It is easy to see that $\overline{\nabla}_X T = \overline{\nabla}_T X$ and X is a Jacobi field along every geodesic $t \to \sigma(t,u)$.

Now, we consider the smooth function Φ on M defined by

$$\Phi(P) = (1/2)\{\mathrm{dist}(Q, f(P))\}^2 \quad \text{for any } P \in M.$$

We wish to estimate the gradient and the Laplacian of Φ.

LEMMA 3. *The following are valid: $\langle \mathrm{grad}\,\Phi, Y\rangle = \langle f_*Y, T\rangle$ and*

$$\Delta\Phi = \langle \tau(f), T\rangle + \sum_i \langle f_*Y_i, \overline{\nabla}_T(f_*Y_i)\rangle.$$

PROOF OF THE LEMMAS. The proofs of Lemmas 1 and 2 are known. Lemma 1 has been proved in [**3**] and it is essentially Omori's Theorem A′ in [**4**]. Lemma 2 can also be deduced from the Rauch comparison theorem given in [**1**, p. 32]. For the proof of Lemma 3 if we put

$$F(\gamma(u)) = \mathrm{dist}(Q, f(\gamma(u))) = \int_0^1 \langle T, T\rangle^{1/2}\, dt$$

then we get $(F \circ \gamma)' = \langle T, T\rangle^{-1/2}\langle f_* Y, T\rangle$. So,

$$\langle \operatorname{grad}\Phi, Y\rangle = (\Phi \circ \gamma)' = (1/2)\{(f \circ \gamma)^2\}' = \langle f_* Y, T\rangle.$$

Also, for the Hessian form $\nabla^2\Phi$ of Φ, we have

$$\begin{aligned}\nabla^2\Phi(Y,Y) &= YY\Phi - (\nabla_Y Y)\Phi = (\Phi\circ\gamma)'' = Y\langle f_* Y, T\rangle \\ &= \langle \overline{\overline{\nabla}}_Y(f_*Y), T\rangle + \langle f_*Y, \overline{\overline{\nabla}}_Y T\rangle = \langle L(Y,Y), T\rangle + \langle f_*Y, \overline{\nabla}_T(f_*Y)\rangle.\end{aligned}$$

So, the second relation of Lemma 3 follows from the above because $\Delta\Phi$ is the trace of $\nabla^2\Phi$.

3. Proof of the theorem. The vector field X can be written $X = X_K + (\langle X, T\rangle/l^2)T$, where X_K is normal to T. If R is the curvature tensor of N we have

$$T^2\langle X, T\rangle = \langle \overline{\nabla}_T\overline{\nabla}_T X, T\rangle = \langle R(T,X)T, T\rangle = 0.$$

So, $T\langle X, T\rangle = h(u)$ and $\langle X, T\rangle = h(u)t$. From the above relations we get, at $t = 1$

$$\langle X_K, X_K\rangle = \langle f_*Y, f_*Y\rangle - \frac{\langle f_*Y, T\rangle^2}{l^2} \tag{1}$$

and

$$\langle \overline{\nabla}_T(f_*Y), f_*Y\rangle = \langle \overline{\nabla}_T X_K, X_K\rangle + \frac{1}{l^2}\langle f_*Y, T\rangle^2. \tag{2}$$

It is easy to see that $X_K = X - (h(u)t/l^2)T$ is a Jacobi field, normal along every geodesic $t \to \sigma(t,u)$ with $X_K(0) = 0$. So, applying Lemma 2 for X_K, we have at $t = 1$

$$\frac{\langle \overline{\nabla}_T X_K, X_K\rangle}{\langle X_K, X_K\rangle} \geq \mu(al) > 0. \tag{3}$$

Now, we pick a point $P_0 \in M$ such that $f(P_0) \neq Q$ and we put $l_0 = \operatorname{dist}(Q, f(P_0))$. Then applying Lemma 1 for the function Φ we have that for any $\varepsilon' > 0$ and $\varepsilon > 0$ there exists a point $P \in M$ depending on P_0 such that

$$\begin{aligned} &\text{(a)} \quad l = \operatorname{dist}(Q, f(P)) \geq \operatorname{dist}(Q, f(P_0)) = l_0;\\ &\text{(b)} \quad \|\operatorname{grad}\Phi\|(P) < \varepsilon';\\ &\text{(c)} \quad \Delta\Phi(P) < \varepsilon.\end{aligned} \tag{4}$$

We are going to use the relations (1)–(4) for this point P. From (1) using Lemma 3 and (4)(a), (b), we deduce now

$$\langle X_K, X_K\rangle > \langle f_*Y, f_*Y\rangle - \frac{\varepsilon'^2}{l_0^2}. \tag{5}$$

When $\mu(al) - 1 \geq 0$ from (2) using (1), (3) and (5), we obtain

$$\begin{aligned}\langle \overline{\nabla}_T(f_*Y), f_*Y\rangle &= \langle f_*Y, f_*Y\rangle - \langle X_K, X_K\rangle + \langle \overline{\nabla}_T X_K, X_K\rangle \\ &\geq \langle f_*Y, f_*Y\rangle - \langle X_K, X_K\rangle + \langle X_K, X_N\rangle\mu(al) \\ &> \langle f_*Y, f_*Y\rangle + (\mu(al) - 1)(\langle f_*Y, f_*Y\rangle > -\varepsilon'^2/l_0^2).\end{aligned}$$

So,

$$\langle \overline{\nabla}_T(f_*Y), f_*Y\rangle > \langle f_*Y, f_*Y\rangle\mu(al) - \varepsilon'^2(\mu(al) - 1)/l_0^2. \tag{6}$$

If $f_*Y_P = 0$ for some $Y_P \in TM$ then the last inequality is trivial. So, applying (6) to an orthonormal basis Y_i, adding and using Lemma 3, we get

$$\Delta\Phi(P) > \langle \tau(f), T\rangle + \sum_i \langle f_*Y_i, f_*Y_i\rangle \mu(al) - \varepsilon'^2(\dim M)(\mu(al) - 1)/l_0^2.$$

Given any positive integer m we choose $\varepsilon = 1/m$ and $\varepsilon'^2 = \varepsilon l_0^2/\dim M$. Using (4)(c) for the points P_m, which correspond to m, the last inequality is converted into

$$\frac{1}{m} > \Delta\Phi(P_m) > \langle \tau(f), T\rangle + e(f)\mu(al_m) - \frac{1}{m}(\mu(al_m) - 1)$$

where $l_m = \operatorname{dist}(Q, f(P_m))$. So, if we suppose that the theorem is not valid, hence $\tau(f) \equiv 0$, we get $e(f) < 1/m$, which contradicts the hypothesis of the theorem.

When $\mu(al) - 1 < 0$, from (2) using (3) and (1), we have

$$\langle \overline{\nabla}_T(f_*Y), f_*Y\rangle \geq \mu(al)\langle f_*Y, f_*Y\rangle.$$

For the rest of the proof we work as before.

An immediate consequence of this theorem is the following corollary.

COROLLARY. *If $f\colon M \to N$ is a harmonic map with $e(f) \geq c > 0$, then $f(M)$ is not contained in a normal ball B_R ($R < \pi/2\sqrt{K_0}$ if $K_0 > 0$).*

If N is a Hadamard manifold (i.e. a simply-connected complete Riemannian manifold with nonpositive sectional curvature) then the corollary says: For any harmonic map $f\colon M \to N$ with $e(f) \geq c > 0$ the image $f(M)$ is unbounded.

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