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A HYPERBOLIC 4-MANIFOLD

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ABSTRACT. There is a regular 4-dimensional polyhedron with 120 dodecahedra as 3-dimensional faces. (Coxeter calls it the "120-cell".) The group of symmetries of this polyhedron is the Coxeter group with diagram:

$$\cdot\overset{5}{\text{—}}\cdot\text{—}\cdot\text{—}\cdot$$

For each pair of opposite 3-dimensional faces of this polyhedron there is a unique reflection in its symmetry group which interchanges them. The result of identifying opposite faces by these reflections is a hyperbolic manifold M^4.

1. Some Coxeter groups. For $0 \leqslant n \leqslant 4$ let G_n denote the Coxeter group of rank $n + 1$ with diagram as indicated below:

$$G_0\ \cdot\ ,\ G_1\ \cdot\overset{5}{\text{—}}\cdot\ ,\ G_2\ \cdot\overset{5}{\text{—}}\cdot\text{—}\cdot\ ,\ G_3\ \cdot\overset{5}{\text{—}}\cdot\text{—}\cdot\text{—}\cdot\ ,\ G_4\ \cdot\overset{5}{\text{—}}\cdot\text{—}\cdot\text{—}\cdot\overset{5}{\text{—}}\cdot$$

Obviously, $G_0 \subset G_1 \subset G_2 \subset G_3 \subset G_4$. The first four of these groups are finite and have canonical representations as subgroups of $O(n + 1)$ (cf. [**1**]). In fact, for $1 \leqslant n \leqslant 3$, G_n is the group of isometries of S^n generated by the orthogonal reflections across the faces of a certain spherical n-simplex Δ^n (a "fundamental chamber"). The group G_4 can be represented as a discrete cocompact subgroup of $O(4,1)$, the group of isometries of hyperbolic 4-space H^4 [**1**, Exercise 15, p. 133]. Its fundamental chamber is a certain hyperbolic 4-simplex Δ^4. For $1 \leqslant n \leqslant 4$ let x_n be a vertex of Δ^n such that the isotropy subgroup of G_n at x_n is G_{n-1}. The translates of Δ^n under G_{n-1} fit together at x_n to give the barycentric subdivision of a convex polyhedron X^n (in S^n if $n \leqslant 3$ or in H^4 if $n = 4$). The translates of X^n under G_n then give a tessellation of S^n ($n \leqslant 3$) or H^4 ($n = 4$) by congruent copies of X^n. The full group of symmetries of this tessellation is G_n. For $1 \leqslant n \leqslant 3$ the convex hull of the vertex set of this tessellation of S^n is a convex polyhedron Y^{n+1} in $\mathbf{R}^{n+1}$. X^1 is a circular arc (of length $2\pi/5$), S^1 is tessellated by 5 copies of it, and Y^2 is a pentagon. X^2 is a spherical pentagon, S^2 is tessellated by 12 copies of it, and Y^3 is a dodecahedron. X^3 is a spherical dodecahedron, S^3 is tessellated by 120 copies of it, and Y^4 is the 4-dimensional regular polyhedron called the "120-cell" in [**3**]. X^4 is a hyperbolic 120-cell, and H^4 is tessellated by an infinite number of copies of it (cf. [**2**]). The orders of these Coxeter groups are as follows: $|G_0| = 2$, $|G_1| = 10$, $|G_2| = 120$, $|G_3| = 14400$ $(= (120)^2)$, $|G_4| = \infty$.

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The group G_n and its fundamental chamber Δ^n can be recovered from the tessellation as follows. Choose a chain of cells in the tessellation: $C^0 \subset C^1 \subset \cdots \subset C^n = X^n$, where C^j has dimension j. Let v_j denote the center of C^j (so that $v_n = x_n$). The vertices $v_0, \ldots, v_n$ span an n-simplex which we can take to be Δ^n. For $0 \leqslant j \leqslant n$ let r_j denote the reflection across the "hyperplane" (i.e., great subsphere if $1 \leqslant n \leqslant 3$ or hyperbolic hyperplane if $n = 4$) supported by the face of Δ^n which is opposite to v_j. The family $(r_j)_{0 \leqslant j \leqslant n}$ is denoted by R_n and called a "fundamental system of reflections" for G_n. Its elements correspond to the nodes of the corresponding diagram (where the nodes are numbered from left to right).

2. The tessellation of S^3 by dodecahedra and its symmetry group. In this section we discuss several facts about G_3. A good general reference for Coxeter groups is [**1**]; in particular, Exercise 12 on p. 231 gives an interesting method of proving some properties of G_3.

Let $\mathscr{D}$ be the set of dodecahedra in the previously mentioned tessellation of S^3. For each $D \in \mathscr{D}$ let

$$\begin{aligned} -D &= \text{the face opposite to } D, \\ v_D &= \text{the center of } D, \\ S_D &= \text{the great 2-sphere orthogonal to } v_D, \\ s_D &= \text{the orthogonal reflection across } S_D. \end{aligned}$$

Clearly, $s_D = s_{D'}$ if and only if $D' = D$ or $-D$.

(2.1) *For each $D \in \mathscr{D}$, the reflection s_D belongs to G_3. Moreover, $(s_D)_{D \in \mathscr{D}}$ is the family of all reflections in G_3.*

The above fact is proved in [**3**, p. 227]. (Alternatively, it follows from the exercise in [**1**] mentioned above.)

Next suppose, as in §1, that $C^0 \subset C^1 \subset C^2 \subset C^3 = X^3$ is a chain of cells and, for $0 \leqslant j \leqslant 3$, v_j is the center of C^j, S_j is the 2-sphere supported by the face of Δ^3 opposite to v_j, and r_j is the orthogonal reflection across S_j. Put $D = C^3$.

(2.2) *For $0 \leqslant j \leqslant 2$ the 2-spheres S_D and S_j make a dihedral angle of $\pi/2$. The 2-spheres S_D and S_3 make a dihedral angle of $2\pi/5$.*

Proof. For $0 \leqslant j \leqslant 2$, S_j contains v_3 $(= v_D)$; hence, S_D and S_j intersect orthogonally. The 2-sphere S_3 is spanned by the spherical pentagon C^2. The circular arc from v_3 to v_2 has length $\pi/10$ [**4**, p. 35]; hence, S_D and S_3 make an angle of $\pi/2 - \pi/10 = 2\pi/5$.

For any two reflections r, r' in a Coxeter group, let $m(r, r')$ denote the order of rr'. As an immediate corollary of (2.2) we have the following fact.

(2.3) *With notation as above put $s = s_D$. For $0 \leqslant j \leqslant 2$, $m(r_j, s) = 2$, while $m(r_3, s) = 5$.*

For $0 \leqslant i \leqslant 3$ let $T_i = (R_3 - \{r_i\}) \cup \{s\}$ (where $R_3 = \{r_0, r_1, r_2, r_3\}$) and let H_i be the subgroup of G_3 generated by T_i.

(2.4) *The pair (H_i, T_i) is a Coxeter system.*

Sketch of Proof. Since H_i is generated by reflections, it is a Coxeter group [**1**, Théorème 1, p. 74]. For $0 \leqslant i \leqslant 3$ let $\hat{H}_i$ be the Coxeter group whose diagram is indicated below.

$$\begin{array}{lccccccccc}
\hat{H}_0 & & & \cdot & \text{---} & \cdot & \text{---} & \cdot & \overset{5}{\text{---}} & \cdot \\
\hat{H}_1 & \cdot & & & & \cdot & \text{---} & \cdot & \overset{5}{\text{---}} & \cdot \\
\hat{H}_2 & \cdot & \overset{5}{\text{---}} & \cdot & & & & \cdot & \overset{5}{\text{---}} & \cdot \\
\hat{H}_3 & \cdot & \overset{5}{\text{---}} & \cdot & \text{---} & \cdot & & & & \cdot
\end{array}$$

From (2.3) it is clear that there is a surjective homomorphism $\hat{H}_i \to H_i$. The meaning of (2.4) is that this map is an isomorphism. This can be proved on a case-by-case basis. The only two case which present any difficulties are H_0 and H_1. The first step is to show that the subgroup H generated by $\{r_2, r_3, s\}$, which is a quotient of G_2, is actually isomorphic to G_2. This follows from the fact that the orientation-preserving subgroup of G_2 is the simple group A_5. Next one can show that H_0 is irreducible and conclude that $\hat{H}_0 \cong H_0$ by arguing that no other irreducible Coxeter group of rank 4 has H as an isotropy subgroup. Similarly, for H_1.

From (2.4) and on p. 20 [**1**, Théorème 2(i)], one can immediately deduce the following fact.

(2.5) *Let T be any proper subset of $R_3 \cup \{s\}$ and let H_T be the subgroup of G_3 generated by T. Then (H_T, T) is a Coxeter system.*

3. A torsion-free subgroup of G_4. Let $\mathscr{D}(X^4)$ denote the set of 3-dimensional faces of the hyperbolic polyhedron X^4 (of course, $\mathscr{D}(X^4)$ can be identified with $\mathscr{D}$). For each $D \in \mathscr{D}(X^4)$, let r_D be the reflection of H^4 across the hyperplane supported by D, let s_D be the reflection of H^4 across the hyperplane through x_4 (the center of X^4) which is orthogonal to the geodesic ray from x_4 to the center of D, and let $t_D = r_D s_D$. It is clear that r_D belongs to G_4, s_D belongs to G_3 (the isotropy subgroup of G_4 at x_4) and, hence, t_D belongs to G_4. Let K denote the subgroup of G_4 generated by the family $(t_D)_{D \in \mathscr{D}(X^4)}$. The transformation t_D takes $-D$ to D, and it takes X^4 to the adjacent 4-cell across D. Hence, X^4 is a fundamental domain for K. Since X^4 is the union of 14400 copies of Δ^4, K has index 14400 in G_4. The orbit space $M^4 = H^4/K$ is obviously the space formed from X^4 by identifying D with $-D$ via s_D for each $D \in \mathscr{D}(X^4)$.

Let $C^0 \subset C^1 \subset C^2 \subset C^3 \subset C^4 = X^4$ be a chain of cells in the tessellation of H^4 and let $R_4 = \{r_0, r_1, r_2, r_3, r_4\}$ be the corresponding set of fundamental reflections for G^4. Define a map from R_4 to G_3 by sending r_i to itself for $0 \leqslant i \leqslant 3$ and by sending r_4 $(= r_D)$ to s_D (where $D = C^3$). It follows from (2.3) that this map extends to a homomorphism $f\colon G_4 \to G_3$ which restricts to the identity on G_3. The kernel of f is K, since K is clearly contained in this kernel and since both groups have the same index in G_4. Thus, G_4 is the semidirect product of G_3 and K. Every finite subgroup of G_4 is conjugate to a subgroup of some "standard subgroup" of G_4 (where "standard subgroup" means a subgroup generated by some proper subset of R_4). It follows from (2.5) that each standard subgroup of G_4 is mapped monomorphically by f. Hence, K is torsion-free. Consequently, K acts freely on H^4, and M^4 is a hyperbolic 4-manifold.

Since K is normal in G_4, the quotient group G_3 acts isometrically on M^4. The orbit space of this action is Δ^4 (the orbit space of G_4 on H^4). The fixed point set of any reflection s_D in G_3 on M^4 is a hyperbolic 3-manifold M^3 obtained by gluing D to the polyhedron formed by intersecting X^4 with the hyperplane in H^4 which is fixed by s_D. Hence, M^3 is 2-sided and connected. It is easy to see that the complement of M^3 in M^4 is connected. Therefore, M^3 represents a nonzero homology class. It follows that the first Betti number of M^4 is $\geqslant 1$. (This Betti number is also $\leqslant 60$, since K is generated by 60 elements and their inverses.)

The Euler characteristic of M^4 is 26. One way to see this is to compute the rational Euler characteristic of G_4, as in [**6**, p. 111], obtaining $\chi(G_4) = 26/14400$.

REMARKS. In [7] Weber and Seifert constructed a hyperbolic 3-manifold by identifying opposite faces of a dodecahedron. In several respects the construction of M^4 seems simpler. The tessellation of H^4 by copies of X^4 is described by Coxeter in [**2**]. The apparently new fact is the existence of the torsion-free subgroup K with X^4 as its fundamental domain. Coxeter also describes two other tessellations of H^4 by regular 120-cells. Their symmetry groups are

$$\bullet\overset{5}{\text{—}}\bullet\text{—}\bullet\text{—}\bullet\text{—}\bullet \quad \text{and} \quad \bullet\overset{5}{\text{—}}\bullet\text{—}\bullet\text{—}\bullet\overset{4}{\text{—}}\bullet$$

These have rational Euler characteristics $1/14400$ and $17/28800$, respectively [**6**, p. 111]. Hence, the second one does not admit a torsion-free subgroup of index 14400; however, the first one quite possibly does. In this vein it might be interesting to find further examples of hyperbolic 4-manifolds formed by identifying faces of a 120-cell. The analogous question in dimension 3 has recently been completely solved in [**5**] with the aid of a machine.

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