ON THE PRESERVATION OF DETERMINACY UNDER CONVOLUTION

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Abstract. In 1959 Devinatz remarked that if $\mu \ast v$ is a determinate measure on the real line, then so are $\mu$ and $v$. It is shown here how this follows from a theorem of M. Riesz, and also how it can be extended to $d$ dimensions. Recently Diaconis raised the question whether the converse is true. We answer this in the negative by producing a determinate measure $v$ on the real line such that $v \ast v$ is indeterminate. The relation to previous work of Heyde and to the condition of Carleman is discussed.

1. The positive result. Let $\mathcal{M}^\ast(\mathbb{R}^d)$ denote the set of positive Borel measures on $\mathbb{R}^d$ having moments of all orders. For $\mu \in \mathcal{M}^\ast(\mathbb{R}^d)$ the moment sequence $s_n(\mu)$ is defined by

$$s_n(\mu) = \int_{\mathbb{R}^d} x^n \, d\mu(x) \quad \text{for } n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d,$$

where $x^n = x_1^{n_1} \cdots x_d^{n_d}$. For $\mu, v \in \mathcal{M}^\ast(\mathbb{R}^d)$ we write $\mu \sim v$ if $\mu$ and $v$ have the same moments, i.e. if $s_n(\mu) = s_n(v)$ for all $n \in \mathbb{N}_0^d$. The equivalence class containing $\mu$ is denoted $[\mu]$, and we recall that $\mu$ is called determinate if $[\mu] = \{\mu\}$, and indeterminate otherwise. In the indeterminate case $[\mu]$ is a convex set, and it is not difficult to see that $[\mu]$ is compact in the weak topology. We shall also need the following result which is a special case of a theorem of Douglas [7], and for $d = 1$ it is usually attributed to Naimark (cf. Akhiezer [1, p. 47]).

1.1. Proposition. Let $\mu \in \mathcal{M}^\ast(\mathbb{R}^d)$. Then $\mu$ is an extreme point of $[\mu]$ if and only if the polynomials are dense in $L^1(\mathbb{R}^d, \mu)$. In particular, the polynomials are dense in $L^1(\mathbb{R}^d, \mu)$ in case $\mu$ is determinate.

For $\mu \in \mathcal{M}^\ast(\mathbb{R}^d)$ the Fourier transform $\hat{\mu} : \mathbb{R}^d \to \mathbb{C}$ is a $C^\infty$-function given by

$$\hat{\mu}(y) = \int_{\mathbb{R}^d} e^{i\langle x, y \rangle} \, d\mu(x), \quad y \in \mathbb{R}^d.$$

The zero set $\{\hat{\mu} = 0\}$ may of course have interior points since it is even possible that $\hat{\mu}$ has compact support. We have, however, the following result:

1.2. Proposition. Assume that $\mu \in \mathcal{M}^\ast(\mathbb{R}^d) \setminus \{0\}$ is an extreme point of $[\mu]$. Then $\{\hat{\mu} = 0\}$ has no interior points, and for $d = 1$ then $\{\hat{\mu} = 0\}$ is a discrete set.
Proof. Assume that $\hat{\mu}$ is identically zero in an open neighbourhood of $y_0 \in \mathbb{R}^d$. Then also

$$D^\alpha \hat{\mu}(y) = i^{\lvert \alpha \rvert} \int_{\mathbb{R}^d} x^\alpha e^{i\langle x, y \rangle} d\mu(x) = 0$$

for all $y$ in this neighbourhood and all $\alpha \in \mathbb{N}^d_0$, and this implies in particular that

$$\int_{\mathbb{R}^d} p(x) e^{i\langle x, y_0 \rangle} d\mu(x) = 0$$

for all polynomials $p$. Since $e^{i\langle x, y_0 \rangle}$ is bounded and $\mu$ is an extreme point of $[\mu]$, we get by Proposition 1.1 that $e^{i\langle x, y_0 \rangle}$ is zero $\mu$-almost everywhere, which is a contradiction.

In the case $d = 1$ we also prove by contradiction that $\{\hat{\mu} = 0\}$ is discrete. In fact, if $\{\hat{\mu} = 0\}$ has an accumulation point $y_0$ we can find a sequence $(y_n)$ of mutually different points from $\{\hat{\mu} = 0\}$ converging to $y_0$, and we may assume that it is either increasing or decreasing. By the mean-value theorem this sequence is separated by a sequence $(\xi_n)$ converging to $y_0$ such that $(\text{Re } \hat{\mu})(\xi_n) = 0$, hence $(\text{Re } \hat{\mu})(y_0) = 0$. Repeated applications of the mean-value theorem show that $(\text{Re } \hat{\mu})(n)(y_0) = 0$ for all $n$ and, similarly, $(\text{Im } \hat{\mu})(n)(y_0) = 0$. Therefore

$$\int (ix)^n e^{ixy_0} d\mu(x) = 0 \quad \text{for all } n \geq 0,$$

and as above this leads to a contradiction. □

If $\mu, \nu \in \mathcal{M}^*(\mathbb{R}^d)$ then the binomial formula implies that $\mu \ast \nu \in \mathcal{M}^*(\mathbb{R}^d)$ and

$$s_n(\mu \ast \nu) = \sum_{0 \leq p \leq n} \binom{n}{p} s_p(\mu)s_{n-p}(\nu), \quad n \in \mathbb{N}^d_0,$$

with the usual notation

$$\binom{n}{p} = \binom{n_1}{p_1} \cdots \binom{n_d}{p_d} \quad \text{for } 0 \leq p_i \leq n_i, \quad i = 1, \ldots, d.$$

The following result is, for $d = 1$, due to Devinatz [5].

1.3. Theorem. Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R}^d)$ be such that $\mu \ast \nu$ is determinate. Then $\mu$ and $\nu$ are determinate.

Proof. Assume that $\mu$ is indeterminate. By the Krein-Milman theorem there are at least two different extreme points $\mu_1, \mu_2$ in $[\mu]$. If $\nu$ is indeterminate we choose an extreme point $\nu_1 \in [\nu]$, and if $\nu$ is determinate we put $\nu_1 = \nu$. Then $\mu_1 \ast \nu_1 \sim \mu_2 \ast \nu_1 \sim \mu \ast \nu$, so $\mu_1 \ast \nu_1 = \mu_2 \ast \nu_1$ because $\mu \ast \nu$ is assumed to be determinate. Then $(\hat{\mu}_1 - \hat{\mu}_2)\hat{\nu}_1 \equiv 0$, but by Proposition 1.2, $\{\hat{\nu}_1 \neq 0\}$ is an open dense subset of $\mathbb{R}^d$, and we get $\hat{\mu}_1 \equiv \hat{\mu}_2$, hence $\mu_1 = \mu_2$, which is a contradiction. □

1.4. Remark. The essential part of the proof of Proposition 1.2 is the essential part of the justification that A. Devinatz had in 1959 for the validity of his remark concerning $\mu \ast \nu$. He has kindly conveyed this to me with permission to present it. In an earlier version of the present paper Theorem 1.3 was proved for $d = 1$ by means of the following result of independent interest:

1.5. Proposition. Let $\mu \in \mathcal{M}^*(\mathbb{R})$ be indeterminate. For every $a \in \mathbb{R} \setminus \{0\}$, $C_a = \{ \hat{\delta}(a) \mid \sigma \in [\mu] \}$ is a compact convex set in $\mathbb{C}$ with nonempty interior.
Proof. Since $[\mu]$ is a compact convex set in the weak topology, it is clear that $C_\mu$ is compact and convex. Let $P$ denote the vector space of real polynomials and consider the positive linear functional $L: P \to \mathbb{R}$ defined by $L(p) = \int p \, d\mu$. If $f: \mathbb{R} \to \mathbb{R}$ is continuous and bounded in absolute value by a polynomial, M. Riesz [13] defined

$$L(f) = \sup \{ L(p) \mid p \in P, p \leq f \},$$

$$\bar{L}(f) = \inf \{ L(p) \mid p \in P, f \leq p \}.$$

Clearly $-\infty < L(f) \leq \bar{L}(f) < \infty$ and $L(f) = \bar{L}(f)$ if $f$ is a polynomial. By the Hahn-Banach theorem it is easy to see that $L(f) < \bar{L}(f)$ for at least one $f \notin P$, since $\mu$ is indeterminate and

$$[L(f), \bar{L}(f)] = \left\{ \int f \, d\sigma \mid \sigma \sim \mu \right\}.$$

A deeper result of M. Riesz [13, §27] asserts that $L(f) < \bar{L}(f)$ for all continuous functions $f \notin P$ which are bounded in absolute value by a polynomial. In particular, $\{ \int \cos(ax - \theta) \, d\sigma(x) \mid \sigma \in [\mu] \}$ is an interval of positive length for each $\theta \in \mathbb{R}$ and this length is the width of $C_\mu$ in the direction $e^{i\theta}$. This shows that $C_\mu$ has interior points. □

It is clear how Theorem 1.3 in case $d = 1$ may be deduced from Proposition 1.5. In fact, if $\mu \in \mathcal{M}^*(\mathbb{R})$ is indeterminate and $\nu \in \mathcal{M}^*(\mathbb{R}) \setminus \{0\}$, we see that $\mu \ast \nu$ is indeterminate by first choosing $a \in \mathbb{R} \setminus \{0\}$ such that $\hat{\nu}(a) \neq 0$, and we then choose $\sigma \sim \mu$ such that $\hat{\sigma}(a) \neq 0$ according to Proposition 1.5. Then $\mu \ast \nu$ and $\sigma \ast \nu$ are equivalent but different.

1.6. Remarks. (a) It is easy to show that if $\mu \in \mathcal{M}^*(\mathbb{R})$ is indeterminate then $\mu \ast \mu$ is indeterminate. Indeed, if $\mu$ is indeterminate then for any real $t$ there is a measure $\sigma_t \sim \mu$ with an atom at $t$ (see [15, p. 60 or Theorem B in 10]). Thus $\sigma_t \ast \sigma_t$ has an atom in $2t$ and is equivalent to $\mu \ast \mu$, which then has to be indeterminate.

(b) As an application of Proposition 1.5 we see that if $\mu \in \mathcal{M}^*(\mathbb{R})$ is indeterminate and symmetric (hence $s_n(\mu) = 0$ for $n$ odd), there exist nonsymmetric measures with the same moments. In fact, if all the measures in $[\mu]$ were symmetric then $C_\mu \subseteq \mathbb{R}$ for all $a \in \mathbb{R}$. This fact has been noticed by Heyde [9].

2. The negative result. The following answers in the negative a question raised by Diaconis and Ylvisaker [6] in a problem arising in statistics. See also [8].

2.1. Theorem. There exists a determinate measure $\nu \in \mathcal{M}^*(\mathbb{R})$ such that $\nu \ast \nu$ is indeterminate.

We shall make use of the Nevanlinna extremal measures (short: N-extremal). If $\mu \in \mathcal{M}^*(\mathbb{R})$ is indeterminate, Nevanlinna gave in 1922 a parametrization of the compact convex set $[\mu]$ by means of the holomorphic functions mapping the upper half-plane into itself. The constant functions $z \mapsto t$, $t \in \mathbb{R} \cup \{\infty\}$, parametrize the N-extremal measures in $[\mu]$. A famous theorem of M. Riesz characterizes the N-extremal measures as the measures $\sigma \in [\mu]$ for which the set of polynomials $P$ is dense in $L^2(\mathbb{R}, \sigma)$. By comparison with Proposition 1.1 we see that the N-extremal measures form a (compact) subset of the extreme points of $[\mu]$. It is known that
N-extremal measures are discrete with countably many atoms. For these results see Shohat-Tamarkin [15, pp. 57–65] or Akhiezer [1, p. 98]. Recent treatments may be found in Landau [10] and Buchwalter and Cassier [4].

We also need the notion of $m$-canonical measures in $[\mu]$. They are parametrized by the holomorphic functions which are rational of degree $m$, so that $0$-canonical measures and N-extremal measures are the same (see Akhiezer [1, p. 115]). Recently Buchwalter and Cassier [3] have characterized the $m$-canonical measures $\sigma \in [\mu]$ as those for which the closure of the polynomials in $L^2(\sigma)$ is of codimension $m$.

**Proof of Theorem 2.1.** Let $\mu \in \mathcal{M}^*(\mathbb{R})$ be an N-extremal indeterminate measure. By the theorem of perturbation [2, Theorem 8], a change of position and size of finitely many atoms of $\mu$ will provide a new N-extremal measure. We may therefore assume that $\mu$ has the form $\mu = \epsilon_0 + \epsilon_1 + \epsilon_2 + \sigma$, where $\sigma$ is a discrete measure with support not containing 0, 1, 2. The measure $\nu = \epsilon_0 + \epsilon_1 + \sigma$ is determinate (cf. [2, Theorem 7]) and $\nu \ast \nu = \mu + \tau$ for some positive measure $\tau$. If $\mu_1 \sim \mu$ then $\mu_1 + \tau \sim \mu + \tau$, which shows that $\nu \ast \nu$ is indeterminate. $\square$

In the following we need an example of a measure with similar properties as above, stated in the following lemma.

2.2. **Lemma.** There exists a determinate probability $\sigma \in \mathcal{M}^*(\mathbb{R})$ such that $x^2\sigma$ is determinate and $\sigma \ast \sigma \ast \sigma$ is indeterminate.

**Proof.** If $\mu$ is N-extremal with no atom at zero then $x^2\mu$ is 1-canonical. In fact, if $k_0 \in L^2(\mu)$ is such that $\int p(x)k_0(x) \, d\mu(x) = p(0)$ for all $p \in P$ (see [2, formula (3)]), then $f(x) = k_0(x)/x \in L^2(x^2\mu)$ and

$$\int p(x)f(x)x^2 \, d\mu(x) = \int xp(x)k_0(x) \, d\mu(x) = 0,$$

showing that $\dim(L^2(x^2\mu)/\overline{P}) \geq 1$. Conversely, if $\phi \in L^2(x^2\mu)$ is orthogonal on $P$, then, defining $\lambda = \int x\phi(x) \, d\mu(x)$, we have for $p \in P$,

$$\int p(x)(x\phi(x) - \lambda k_0(x)) \, d\mu(x) = \int (p(x) - p(0))x\phi(x) \, d\mu(x)$$

$$= \int q(x)\phi(x)x^2 \, d\mu(x) = 0$$

because $q(x) := x^{-1}(p(x) - p(0)) \in P$. This shows that $x\phi(x) = \lambda k_0(x)$ $\mu$-a.e. so that finally $\dim(L^2(x^2\mu)/\overline{P}) = 1$.

Let $\mu = \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \sigma$ be an N-extremal measure where the support of $\sigma$ is different from \{0, 1, 2, 3\}. Then $\nu = \epsilon_0 + \epsilon_1 + \sigma$ is determinate and $\nu \ast \nu \ast \nu$ is indeterminate since it majorizes $\mu$. For any $a \in \mathbb{R}$ we have that $\nu \ast \epsilon_a$ is determinate and $(\nu \ast \epsilon_a) \ast \nu$ is indeterminate, and we choose $a$ so small that $0 \notin \text{supp}(\mu \ast \epsilon_a)$. Then $x^2(\mu \ast \epsilon_a)$ is 1-canonical, so by the theorem of subtraction by Buchwalter and Cassier (see [3, Théorème 4]), $x^2(\nu \ast \epsilon_a)$ is determinate. The measure $\sigma = \nu(\mathbb{R})^{-1}(\nu \ast \epsilon_a)$ has the desired properties. $\square$

2.3. **Applications.** (a) The proof of Theorem 2.1 also shows that the N-extremal measure $\mu$ is the sum of two determinate measures $\epsilon_0$ and $\nu$. This result was used in
Petersen [12] to construct a determinate measure in $\mathbb{R}^2$ with indeterminate marginal distributions.

(b) There are probabilities $\mu$ and $\sigma$, say on $[0, \infty[, \text{ with } \mu \text{ indeterminate, } \sigma \text{ determinate and yet } s_n(\mu) \leq s_n(\sigma) \text{ for all } n > 0. \text{ In fact if } \mu \text{ is an N-extremal probability on } [0, \infty[ \text{ of the form } \mu = \frac{1}{2} \delta_0 + \frac{1}{2} \sigma \text{ and } 0 \text{ is not an atom of } \sigma, \text{ then } s_n(\mu) = \frac{1}{2} s_n(\sigma) \text{ for } n > 1.$

(c) The construction of Lemma 2.2 can be used to produce a counterexample to a statement of Heyde [9, Corollary 2], which in corrected form says: There exists an infinitely divisible and indeterminate probability $\mu$ such that $\mu$ is the only infinitely divisible measure in $[\mu]$.

To see this we recall Kolmogorov’s canonical representation of an infinitely divisible probability $\mu \in \mathcal{M}^*(\mathbb{R})$:

$$\log \mu(y) = icy + \int_{-\infty}^{\infty} \left( e^{ixy} - 1 - ixy \right) x^{-2} d\lambda(x),$$

where $c \in \mathbb{R}$ and $\lambda \in \mathcal{M}^*(\mathbb{R})$ are uniquely determined by $\mu$ (cf., e.g., Lukacs [11, p. 90]). Note that

$$\frac{1}{n!} \left[ \frac{d^n}{dx^n} \log \mu(y) \right]_{y=0} = \left\{ s_n(\lambda), \quad n \geq 2. \right.$$  

If $\lambda_1 \sim \lambda_2$ are different we may use (1) (with $c = 0$) to define two different infinitely divisible probabilities $\mu_1, \mu_2 \in \mathcal{M}^*(\mathbb{R})$, and from (2) it is easily seen that $\mu_1 \sim \mu_2$.

Conversely, if $\mu_1 \sim \mu_2$ are different infinitely divisible measures, then the corresponding measures $\lambda_1, \lambda_2$ given by (1) are different and satisfy $\lambda_1 \sim \lambda_2$ by (2). This shows that if $\mu$ is determinate then so is $\lambda$. We give an example of an indeterminate infinitely divisible $\mu$, for which $\lambda$ is determinate, and get, as asserted, that $\mu$ is the only infinitely divisible measure in $[\mu]$.

Let $\sigma$ be the measure constructed in Lemma 2.2 and define

$$\mu = e^{-1} \left( \delta_0 + \sigma + \frac{1}{2} \sigma * \sigma + \cdots + \frac{1}{n!} \sigma * \sigma + \cdots \right).$$

Then $\mu$ is infinitely divisible and majorizes a multiple of $\sigma^3$, hence indeterminate. Since $\log \mu = \delta - 1$, we see from (1) that $\lambda = x^2 \sigma$ (and $c = \int x d\sigma(x)$), so $\lambda$ is determinate.

3. Carleman’s condition. A well-known sufficient condition for $\mu \in \mathcal{M}^*(\mathbb{R})$ to be determinate is the Carleman condition

$$\sum_{n=0}^{\infty} (s_{2n}(\mu))^{-1/2n} = \infty.$$  

As shown by the measures in 2.3(b), condition (C) is not necessary.

It is worth noticing that San Juan in 1936 (cf. [14]) showed that Stieltjes’ famous indeterminate measure $\mu$ with density $\exp(-x^{1/4})$ on $[0, \infty[$ can be decomposed as $\mu_1 + \mu_2$ with $\mu_1 \perp \mu_2$ such that (C) is satisfied for both $\mu_1$ and $\mu_2$. His construction applies to any indeterminate measure on $[0, \infty[$.
3.1. PROPOSITION. Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$. If (C) holds for $\mu * \nu$ then it also holds for $\mu$ and $\nu$.

It is possible that (C) holds for $\mu$ and $\nu$ but that $\mu * \nu$ is indeterminate.

PROOF. The first part of the result is due to Devinatz [5], and the last part follows from San Juan's construction. In fact, if $\mu = \delta_0 + \mu_1$, $\nu = \delta_0 + \mu_2$ then (C) holds for $\mu$ and $\nu$, but $\mu * \nu$ majorizes the Stieltjes measure and is therefore indeterminate. \Box

3.2. PROPOSITION. If (C) holds for $\mu \in \mathcal{M}^*(\mathbb{R})$ then it also holds for $\mu * \mu$.

PROOF. We may assume $\mu(\mathbb{R}) = 1$. By Hölder's inequality we have, for $0 < k < 2n$,

$$|s_k(\mu)| \leq s_{2n}(\mu)^{k/2n},$$

and, hence,

$$s_{2n}(\mu * \mu) = \sum_{k=0}^{2n} \binom{2n}{k} s_k(\mu) s_{2n-k}(\mu) \leq \sum_{k=0}^{2n} \binom{2n}{k} (s_{2n}(\mu))^{k/2n} (s_{2n}(\mu))^{(2n-k)/2n} = (2s_{2n}(\mu))^{1/2n},$$

which shows the assertion. \Box

3.3. COROLLARY. Let $\mu \in \mathcal{M}^*(\mathbb{R})$ be an infinitely divisible probability and let $(\mu_t)_{t>0}$ be the convolution semigroup such that $\mu_t = \hat{\mu}^t$ for $t > 0$. If (C) holds for one of the measures $\mu_t$, it holds for all.

PROOF. If (C) holds for $\mu_{t_0}$, it holds for $\mu_t$ for $t = 2^n t_0$, $n = 1, 2, \ldots$, and hence for all $t$ by Proposition 3.1. \Box

3.4. REMARK. Let $(\mu_t)_{t>0}$ be a convolution semigroup as above. It follows by Theorem 1.3 that if $\mu_{t_0}$ is determinate then so is $\mu_t$ for $t \leq t_0$. If $\lambda$ is the measure corresponding to $\mu_1$ in (1) then $t\lambda$ corresponds to $\mu_t$. If $\lambda$ is indeterminate then all measures in $(\mu_t)_{t>0}$ are indeterminate. We have not been able to decide if it is possible that $\mu_t$ is determinate for $t < t_0 < \infty$ and $\mu_t$ is indeterminate for $t > t_0$. If such an example exists then $\lambda$ must be determinate.

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