CONTINUITY OF HOMOMORPHISMS ON A BAIRE GROUP

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Abstract. A pointwise converging sequence of continuous homomorphisms is equicontinuous.

Banach [1, Satz 7, p. 108] proved that the pointwise limit of a sequence of continuous homomorphisms on a Baire group is continuous. This was refined by Pettis [5, Corollary 2.1, p. 297] who showed that a pointwise converging sequence of continuous homomorphisms on a Baire group is equicontinuous—which entails the continuity of the limit. His proof is dense, appealing to extraneous concepts and external previously proved results and imposing on the reader a modification of the major portion of the preceding theorem’s proof. We present a direct, self-contained, and uncluttered proof of a somewhat more general result: A sequence of continuous homomorphisms which is pointwise right- (or left-) Cauchy on a nonmeager subset of a topological group is equicontinuous; whence, its limit is continuous wherever it exists.

Let \( f_n \) be a sequence of continuous functions from a topological space \( X \) to a uniform space \( Y \) which is pointwise Cauchy, and let \( V \) be an entourage of \( Y \) closed in \( Y \times Y \): the set of \( x \) at which \( (f nx, f nx) \in V \) is closed in \( X \) and therefore so is the intersection of these sets for \( m \geq n \). These intersections increase with \( n \), and the content of pointwise Cauchyness on a set \( C \) is just that their union contains \( C \). If \( C \) is nonmeager, one of these must have nonvoid interior and thus may, if \( X \) is a group, be written as a translate \( Ux \) of a neighborhood of the identity \( e \). Taking also \( Y \) to be a group uniformized by its right translates and the \( f_n \) to be homomorphisms, this yields \( f nu = f nx(f nx)^{-1}f nx(f nx) = V u V^{-1} \) for \( u \in U \) and \( m \geq n \). Since \( f_n \) is continuous at \( e \), it follows that \( f nu \in VV^{-1} \) for all \( m \geq n \) on some possibly smaller \( U \). Since there are only finitely many preceding \( m \), this entails the equicontinuity of the \( f_n \) and the continuity of their limit.

Pettis postulates pointwise convergence of the sequence of homomorphisms (rather than their one-sided Cauchyness) on a nonmeager subset with the Baire property, that is, one which differs by a meager set from an open set (“almost open” in Bourbaki). The presence of such a subset entails that the group is Baire (which we do
not require); on the other hand, there are nonmeager subsets of the real line, which are not almost open (Bourbaki [2, IX.5, Exercise 27]).

A different generalization of Banach's theorem is offered in [3].

REFERENCES


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