PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 2, February 1985

CONTINUITY OF HOMOMORPHISMS ON A BAIRE GROUP

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ABSTRACT. A pointwise converging sequence of continuous homomorphisms is equicontinuous.

Banach [**1**, Satz 7, p. 108] proved that the pointwise limit of a sequence of continuous homomorphisms on a Baire group is continuous. This was refined by Pettis [**5**, Corollary 2.1, p. 297] who showed that a pointwise converging sequence of continuous homomorphisms on a Baire group is equicontinuous—which entails the continuity of the limit. His proof is dense, appealing to extraneous concepts and external previously proved results and imposing on the reader a modification of the major portion of the preceding theorem's proof. We present a direct, self-contained, and uncluttered proof of a somewhat more general result: *A sequence of continuous homomorphisms which is pointwise right-* (*or left-*) *Cauchy on a nonmeager subset of a topological group is equicontinuous*; *whence*, *its limit is continuous wherever it exists*.

Let f_n be a sequence of continuous functions from a topological space X to a uniform space Y which is pointwise Cauchy, and let V be an entourage of Y closed in $Y \times Y$: the set of x at which $(f_m x, f_n x) \in V$ is closed in X and therefore so is the intersection of these sets for $m \geqslant n$. These intersections increase with n, and the content of pointwise Cauchyness on a set C is just that their union contains C. If C is nonmeager, one of these must have nonvoid interior and thus may, if X is a group, be written as a translate Ux of a neighborhood of the identity e. Taking also Y to be a group uniformized by its right translates and the f_n to be homomorphisms, this yields $f_m u = f_m ux(f_n ux)^{-1} f_n u(f_m x f_n x^{-1})^{-1} \in V f_n u V^{-1}$ for $u \in U$ and $m \geqslant n$. Since f_n is continuous at e, it follows that $f_m u \in VVV^{-1}$ for all $m \geqslant n$ on some possibly smaller U.[2] Since there are only finitely many preceding m, this entails the equicontinuity of the f_n and the continuity of their limit.

Pettis postulates pointwise convergence of the sequence of homomorphisms (rather than their one-sided Cauchyness) on a nonmeager subset with the Baire property, that is, one which differs by a meager set from an open set ("almost open" in Bourbaki). The presence of such a subset entails that the group is Baire (which we do

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Received by the editors November 20, 1983 and, in revised form, March 26, 1984.
1980 *Mathematics Subject Classification*. Primary 22A10.
Key words and phrases. Equicontinuity, Baire, topological group.
[1] Both authors benefited from NSERC Operating Grants.

[2] Thus, $f_n u$ converges to e in Y along the product filter, Fréchet in $\mathbf{N}$ by neighborhoods of e in X; that is, f_n converges continuously at e.

not require); on the other hand, there are nonmeager subsets of the real line, which are not almost open (Bourbaki [**2**, IX.5, Exercise 27]).

A different generalization of Banach's theorem is offered in [**3**].

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