MAXIMAL COMPACT NORMAL SUBGROUPS
AND PRO-LIE GROUPS
R. W. BAGLEY AND T. S. WU

ABSTRACT. We are concerned with conditions under which a locally compact

group $G$ has a maximal compact normal subgroup $K$ and whether or not $G/K$

is a Lie group. If $G$ has small compact normal subgroups $K$ such that $G/K$

is a Lie group, then $G$ is pro-Lie. If in $G$ there is a collection of closed normal

subgroups $\{H_\alpha\}$ such that $\bigcap H_\alpha = e$ and $G/H_\alpha$ is a Lie group for each $\alpha$, then

$G$ is a residual Lie group. We determine conditions under which a residual Lie
group is pro-Lie and give an example of a residual Lie group which is not

embeddable in a pro-Lie group.

DEFINITION 1. A locally compact topological group $G$ is an $L$-group if, for each

neighborhood $U$ of the identity and compact subset $C$ of $G$, there is a neighborhood

$V$ such that $gH_\alpha g^{-1} \cap C \subset U$ for every $g \in G$ and subgroup $H \subset V$.

DEFINITION 2. A topological group $G$ is pro-Lie if $G$ has small compact normal

subgroups $K$ such that $G/K$ is a Lie group.

DEFINITION 3. A locally compact group $G$ is a residual Lie group if there is a

collection of closed normal subgroups $\{H_\alpha\}$ such that $\bigcap H_\alpha = e$ and $G/H_\alpha$ is a Lie

group for each $\alpha$.

It is easy to see that a residual Lie group is an $L$-group [1, Theorem 1.3 and its

Corollary]. In this paper we obtain the following: If $G$ is an $L$-group and has

an open normal subgroup $G_1$ such that $G_1/G_0$ is compact, then $G$ is pro-Lie. There

exists a locally compact group $G$ with a maximal compact normal subgroup $K$ such

that $G/K$ is not a Lie group. A residual Lie group is not necessarily embeddable

by a continuous isomorphism in a pro-Lie group. A generalized FC-group has a

maximal compact normal subgroup.

PROPOSITION 1. For a locally compact group $G$, the following are equivalent:

(1) $G$ has a compact normal subgroup $K$ such that $G/K$ is a Lie group.

(2) $G$ has a compact normal subgroup $K$ such that $G/K$ is locally connected.

(3) $G$ has an open normal subgroup $G_1$ such that $G_1/G_0$ is compact.

We denote the connected component of the identity of $G$ by $G_0$.

PROOF. Obviously (1) implies (2). We assume that $K$ is compact normal and

$G/K$ is locally connected. Thus $(G/K)_0 = G_0/K$ is open in $G/K$. It follows that

$KG_0 = G_1$ is an open normal subgroup of $G$ and $G_1/G_0$ is compact. We complete

the proof by showing that (3) implies (1). Assuming (3) we have a maximal compact

normal subgroup $K$ of $G_1$ and $G_1/K$ is a Lie group [1, Proposition 2.6]. Since $K$

is maximal in $G_1$ and $G_1/K$ is a Lie group, $K$ is normal in $G$. Since $G/G_1$ is discrete

and $G_1/K$ is a Lie group, $G/K$ is a Lie group.

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THEOREM 2. For a locally compact group $G$, the following are equivalent:

(1) $G$ is pro-Lie.

(2) $G$ is an L-group and $G/G_0$ is pro-Lie.

(3) $G$ is an L-group and has a compact normal subgroup $K$ such that $G/K$ is locally connected.

(4) $G$ is an L-group and has an open normal subgroup $G_1$ such that $G_1/G_0$ is compact.

PROOF. Obviously (1) implies (3). The reverse implication follows from Proposition 1 and [1, Theorem 1.2 and Corollary 2]. Also by Proposition 1, (3) and (4) are equivalent. Since (1) obviously implies (2) we can complete the proof by showing that (2) implies (4). This follows immediately from [1, Proposition 1.7 and Corollary 2].

If $G$ is a locally compact group and $K$ is a maximal compact normal subgroup, then there are some obvious cases when $G/K$ is a Lie group. For example:

(1) If there is an open normal subgroup $G_1$ such that $G_1/G_0$ is compact, then $G/K$ is a Lie group. Now $G_1$ has a maximal compact normal subgroup $K_1$ such that $G_1/K_1$ is a Lie group. Also $K_1$ is normal in $G$; thus, $G/K_1$ is a Lie group. It follows that, if $K$ is maximal compact normal in $G$, then $G/K$ is a Lie group.

(2) If $G/H$ is locally connected for some compact normal subgroup $H$, then $G/K$ is a Lie group.

(3) If $G/K$ is pro-Lie, then it is a Lie group.

By the semidirect product $H \times_\eta K$ of two groups $H$ and $K$ we mean the group determined by a homomorphism $\eta: K \to \mathcal{A}(H)$, the automorphism group of $H$, with the group operation in $H \times_\eta K$ defined by $(h, k)(h_1, k_1) = (h\eta(k)(h_1), kk_1)$.

EXAMPLE 1. Let $G = \bigoplus_{i=1}^{\infty} K_i \times \bigoplus_{i=1}^{\infty} K_i \times Z$, where $K_i = \{0, 1\}$ and the integer $n \in Z$ shifts coordinates $n$ places to the left. We assume $\sum K_i$ has the discrete topology and $\prod K_i$ has the product topology. It is easy to see that any nontrivial invariant subset $C$ of $G$ contains elements $(k(n), z^j)$, where the $n$th coordinate of $k(n)$ is 1 for infinitely many integers $n$, and $j$ is some fixed integer. Thus $C$ is not compact. It follows that the maximal compact normal subgroup of $G$ is the identity. Thus we have a compactly generated group with maximal compact normal subgroup $K$ such that $G/K$ is not a Lie group.

There are many examples of topological groups without maximal compact normal subgroups; for example, any discrete group with no maximal finite subgroup.

EXAMPLE 2. We give another example of a locally compact group $G$ which is not a Lie group and in which the identity is the maximal compact normal subgroup. Let $G = \bigoplus_{i=1}^{\infty} H_i \times_\eta \bigoplus_{i=0}^{\infty} K_i$, where $H_i = Z$, the additive group of integers, and $K_i = Z_2$, the multiplicative two-element group. The automorphism $\eta(k)$ of $\bigoplus H_i$ is defined as $\eta(k)(h_i) = k_i h_i$. It is easy to see that if $\sum H_i$ has the discrete topology and $\prod K_i$ the product topology, $G$ has the desired properties.

DEFINITION. A locally compact group $G$ is a generalized FC-group if $G = A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} = e$, where each $A_i$ is a closed normal subgroup of $G$ and $A_i/A_{i+1}$ is a compactly generated FC-group.

We note that by [2, Corollary 3.9], every compactly generated FC-group has a compact normal subgroup such that the corresponding factor group is of the form $V \times Z^k$. 

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THEOREM 3. If \( G \) is a locally compact generalized FC-group, then \( G \) has a maximal compact normal subgroup.

PROOF. We proceed by induction, noting that, for \( n = 1 \), the conclusion follows from the corollary referred to above. Let \( G = A_1 \supset A_2 \supset \cdots \supset A_{n+1} = e \), where each \( A_i/A_{i+1} \) is a compactly generated FC-group. We assume that every generalized FC-group with sequence \( \{A_i\} \) of shorter length has a maximal compact normal subgroup. Thus \( G/A_n \) has a maximal compact normal subgroup \( M \) and \( A_n \) has a maximal compact normal subgroup \( K \). Let \( N = \pi^{-1}(M) \), where \( \pi \) is the canonical mapping of \( G \) onto \( G/A_n \). Again, using [2, Corollary 3.9], we let \( Q = Z(A_n/K, N/K) \) and note that \( Q \) is a compactly generated central group. Thus \( Q \) has a maximal compact normal subgroup. We let \( P \) be the inverse image of this subgroup under the canonical mapping of \( N \) onto \( N/K \). We show that \( P \) is the maximal compact normal subgroup of \( G \). Let \( L \) be any compact normal subgroup of \( G \). Then \( \pi(L) \subset M \) and \( L \subset N \). Also \( L/K \cap A_n/K = \{e\} \) since \( A_n/K \cong V \times Z^k \). If follows that \( L/K \subset Q \) and \( L \subset P \) as desired.

DEFINITION 4. A topological group \( G \) is an \( N \)-group if for every neighborhood \( U \) of the identity and compact set \( C \) of \( G \), there is a neighborhood \( V \) of the identity such that \( gVg^{-1} \subset U \cup (G-C) \) for all \( g \in G \). A group \( G \) is a SIN-group if \( G \) has small neighborhoods of the identity which are invariant under the inner automorphisms of \( G \).

It is easy to see that, if a topological group \( G \) can be embedded in an SIN-group, then \( G \) is an \( N \)-group.

EXAMPLE 3. We construct an example of an \( N \)-group which is residual Lie and which cannot be embedded in a pro-Lie group. We begin with the group \( F \) of \( 2 \times 2 \) matrices of determinant equal to 1. Let \( H \) be the subgroup of \( F \) of matrices of the form \( \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \). If \( N_0 \) is a normal subgroup of \( F \times F \) which contains all elements of the form \( [A, A^{-1}] \), then \( N_0 \) contains all pairs of the form \([I, C]\), where \( I \) is the identity and \( C \) is an element of \( H \). To show this, let \( A = \begin{pmatrix} a/y & 1/a \\ 0 & y \end{pmatrix} \) and \( B = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix} \). Then

\[
[AB, A^{-1}B^{-1}] = \begin{pmatrix}
1/y & a \\
0 & y^{-1}a
\end{pmatrix},
\begin{pmatrix}
y & 1/y \\
0 & 1/y
\end{pmatrix}
\]

is in \( N_0 \). Thus, if

\[
A = \begin{pmatrix}
a^2 - 1/ca & a \\
0 & ca/a^2 - 1
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
c/a^2 - 1 & -1/a \\
0 & a^2 - 1/ca
\end{pmatrix},
\]

then \([A, B] \) is in \( N_0 \). It follows that \([I, \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}] \) is in \( N_0 \) as desired, since \([I, \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}] = [A, B][A^{-1}, A]\).

If \( c > 0 \) and \( C = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \), then \([I, C] \) is the conjugate of \([I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}] \) by \([A^{-1}, A] \), where

\[
A = \begin{pmatrix}
\sqrt{c} & 1 \\
0 & 1/\sqrt{c}
\end{pmatrix}.
\]

If \( c < 0 \), then \([I, C] \) is the conjugate of \([I, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}] \) by \([A^{-1}, A] \), where

\[
A = \begin{pmatrix}
\sqrt{-c} & -1 \\
0 & 1/\sqrt{-c}
\end{pmatrix}.
\]

Thus, \( N_1 = I \times H \) has only three conjugacy classes relative to \( N_0 \). We show that \( N_0 \) cannot be embedded (isomorphically) in a compact group.
LEMMA. If G is a discrete group and H is an infinite subgroup which has only finitely many conjugacy classes relative to G, then G cannot be embedded in a compact group.

PROOF. Suppose there exists an isomorphism Q: G → K, a compact group. If \( H_0 \) is a conjugacy class of H such that \( H_0 \neq \{e\} \), then there is a neighborhood \( V \) of the identity of K such that \( Q(H_0) \cap V = \emptyset \). Because, if \( Q(H_0) \) intersects every neighborhood of the identity of K, then for any neighborhood W there is a neighborhood V of the identity of K such that \( gVg^{-1} \subset W \). If \( Q(H_0) \cap V \neq \emptyset \), then \( Q(H_0) \subset W \). Consequently \( Q(H_0) = e \) which is a contradiction. Since H has only finitely many conjugacy classes, there is a neighborhood U of the identity of K such that \( Q(H) \cap U = e \). Thus \( Q(H) \) is discrete which contradicts the fact that K is compact.

We are ready to construct the example referred to above. Let \( F_i = F \times F \), where F is the group of 2 x 2 matrices referred to above and \( K_i = Z_2 \) for each positive integer i. Let \( G = \sum F_i \times_{\eta} \prod K_i \), where \( \sum F_i \) is the direct sum with discrete topology, \( \prod K_i \) is the direct product with product topology, and \( \eta \) is defined as follows: If the ith coordinate of \( k \in \prod K_i \) is 1, then the ith coordinate of \( \eta(k)(f) \) is \( (f_2, f_1) \), where the ith coordinate of \( f \) is \( (f_1, f_2) \). If the ith coordinate of \( k \) is 0, then, of course, \( \eta(k) \) does not alter the ith coordinate of \( f \in \sum F_i \).

We note that any open normal subgroup of G contains an isomorphic image of a subgroup \( N_0 \) of \( F \times F \) such that \( [A, A^{-1}] \in N_0 \) for each \( A \in F \). To see this we note that, if \( k \in \prod K_i \) has ith coordinate 1 and all other coordinates 0, and \( f \in \sum F_i \) has ith coordinate \( [A, I] \), and all other coordinates \( [I, I] \), then \( (f, 0) \in \sum F_i \times_{\eta} \prod K_i \), and \( (f, 0)(I, k)(f, 0)^{-1} = (f', k) \), where \( f' \) has ith coordinate \( [A, A^{-1}] \) and all other coordinates \( [I, I] \), where I here is the identity of F.

To show that G cannot be embedded in a pro-Lie group, assume there exists a continuous isomorphism \( Q: F \to G_1 \) and \( G_1 \) is pro-Lie. Let K be a normal compact subgroup of \( G_1 \) such that \( G_1/K \) is a Lie group. Then \( G/Q^{-1}(K) \) is a Lie group, hence discrete since G is totally disconnected. It follows that \( Q^{-1}(K) \) is an open normal subgroup of G. This contradicts the Lemma. Thus, the existence of the continuous isomorphism \( Q \) is contradicted.

Since a locally compact SIN-group is pro-Lie [2, Theorem 2.11], it follows from the above argument that G cannot be embedded in a locally compact SIN-group.

In conclusion we see that G is an N-group and residual Lie. Actually G can be embedded in \( \sum F_i \times_{\eta} \prod K_i \), where \( \sum F_i \times \prod K_i \) has the relative product topology of \( \prod F_i \times \prod K_i \) which is SIN but, of course, not locally compact.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124

DEPARTMENT OF MATHEMATICS AND STATISTICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106