MAXIMAL COMPACT NORMAL SUBGROUPS
AND PRO-LIE GROUPS
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ABSTRACT. We are concerned with conditions under which a locally compact


group \( G \) has a maximal compact normal subgroup \( K \) and whether or not \( G/K \)


is a Lie group. If \( G \) has small compact normal subgroups \( K \) such that \( G/K \)


is a Lie group, then \( G \) is pro-Lie. If in \( G \) there is a collection of closed normal


subgroups \( \{H_\alpha\} \) such that \( \bigcap H_\alpha = e \) and \( G/H_\alpha \) is a Lie group for each \( \alpha \), then

\( G \) is a residual Lie group. We determine conditions under which a residual Lie


group is pro-Lie and give an example of a residual Lie group which is not


embeddable in a pro-Lie group.


DEFINITION 1. A locally compact topological group \( G \) is an \( L \)-group if, for each


neighborhood \( U \) of the identity and compact subset \( C \) of \( G \), there is a neighborhood


\( V \) such that \( gHg^{-1} \cap C \subset U \) for every \( g \in G \) and subgroup \( H \subset V \).


DEFINITION 2. A topological group \( G \) is pro-Lie if \( G \) has small compact normal


subgroups \( K \) such that \( G/K \) is a Lie group.


DEFINITION 3. A locally compact group \( G \) is a residual Lie group if there is a


collection of closed normal subgroups \( \{H_\alpha\} \) such that \( \bigcap H_\alpha = e \) and \( G/H_\alpha \) is a Lie


group for each \( \alpha \).


It is easy to see that a residual Lie group is an \( L \)-group [1, Theorem 1.3 and its


Corollary]. In this paper we obtain the following: If \( G \) is an \( L \)-group and has an


open normal subgroup \( G_1 \) such that \( G_1/G_0 \) is compact, then \( G \) is pro-Lie. There


exists a locally compact group \( G \) with a maximal compact normal subgroup \( K \) such


that \( G/K \) is not a Lie group. A residual Lie group is not necessarily embeddable


by a continuous isomorphism in a pro-Lie group. A generalized FC-group has a


maximal compact normal subgroup.


PROPOSITION 1. For a locally compact group \( G \), the following are equivalent:

(1) \( G \) has a compact normal subgroup \( K \) such that \( G/K \) is a Lie group.

(2) \( G \) has a compact normal subgroup \( K \) such that \( G/K \) is locally connected.

(3) \( G \) has an open normal subgroup \( G_1 \) such that \( G_1/G_0 \) is compact.

We denote the connected component of the identity of \( G \) by \( G_0 \).


PROOF. Obviously (1) implies (2). We assume that \( K \) is compact normal and

\( G/K \) is locally connected. Thus \( (G/K)_0 = G_0/K \) is open in \( G/K \). It follows that

\( K/G_0 = G_1 \) is an open normal subgroup of \( G \) and \( G_1/G_0 \) is compact. We complete

the proof by showing that (3) implies (1). Assuming (3) we have a maximal compact

normal subgroup \( K \) of \( G_1 \) and \( G_1/K \) is a Lie group [1, Proposition 2.6]. Since \( K \)

is maximal in \( G_1 \) and \( G_1/K \) is normal in \( G \), \( K \) is normal in \( G \). Since \( G/G_1 \) is discrete

and \( G_1/K \) is a Lie group, \( G/K \) is a Lie group.


Received by the editors January 11, 1984.

1980 Mathematics Subject Classification. Primary 22D05.
THEOREM 2. For a locally compact group $G$, the following are equivalent:

1. $G$ is pro-Lie.
2. $G$ is an $L$-group and $G/G_0$ is pro-Lie.
3. $G$ is an $L$-group and has a compact normal subgroup $K$ such that $G/K$ is locally connected.
4. $G$ is an $L$-group and has an open normal subgroup $G_1$ such that $G_1/G_0$ is compact.

PROOF. Obviously (1) implies (3). The reverse implication follows from Proposition 1 and [1, Theorem 1.2 and Corollary 2]. Also by Proposition 1, (3) and (4) are equivalent. Since (1) obviously implies (2) we can complete the proof by showing that (2) implies (4). This follows immediately from [1, Proposition 1.7 and Corollary 2].

If $G$ is a locally compact group and $K$ is a maximal compact normal subgroup, then there are some obvious cases when $G/K$ is a Lie group. For example:

1. If there is an open normal subgroup $G_1$ such that $G_1/G_0$ is compact, then $G/K$ is a Lie group. Now $G_1$ has a maximal compact normal subgroup $K_1$ such that $G_1/K_1$ is a Lie group. Also $K_1$ is normal in $G$; thus, $G/K_1$ is a Lie group. It follows that, if $K$ is maximal compact normal in $G$, then $G/K$ is a Lie group.
2. If $G/H$ is locally connected for some compact normal subgroup $H$, then $G/K$ is a Lie group.
3. If $G/K$ is pro-Lie, then it is a Lie group.

By the semidirect product $H \rtimes K$ of two groups $H$ and $K$ we mean the group determined by a homomorphism $\eta: K \to \mathcal{A}(H)$, the automorphism group of $H$, with the group operation in $H \rtimes K$ defined by $(h, k)(h_1, k_1) = (h\eta(k)(h_1), kk_1)$.

EXAMPLE 1. Let $G = \bigoplus_{n=-\infty}^{\infty} K_i \times (\prod_{i=1}^{\infty} K_i) \times Z$, where $K_i = \{0, 1\}$ and the integer $n \in Z$ shifts coordinates $n$ places to the left. We assume $\bigcup K_i$ has the discrete topology and $\prod K_i$ has the product topology. It is easy to see that any nontrivial invariant subset $C$ of $G$ contains elements $(k(n), z^j)$, where the $n$th coordinate of $k(n)$ is 1 for infinitely many integers $n$, and $j$ is some fixed integer. Thus $C$ is not compact. It follows that the maximal compact normal subgroup of $G$ is the identity. Thus we have a compactly generated group with maximal compact normal subgroup $K$ such that $G/K$ is not a Lie group.

There are many examples of topological groups without maximal compact normal subgroups; for example, any discrete group with no maximal finite subgroup.

EXAMPLE 2. We give another example of a locally compact group $G$ which is not a Lie group and in which the identity is the maximal compact normal subgroup. Let $G = \bigoplus_{i=1}^{\infty} H_i \times \eta \prod_{i=0}^{\infty} K_i$, where $H_i = Z$, the additive group of integers, and $K_i = Z_2$, the multiplicative two-element group. The automorphism $\eta(k)$ of $\bigoplus H_i$ is defined as $\eta(k)(h_i) = k_i h_i$. It is easy to see that if $\bigcup H_i$ has the discrete topology and $\prod K_i$ the product topology, $G$ has the desired properties.

DEFINITION. A locally compact group $G$ is a generalized $FC$-group if $G = A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} = e$, where each $A_i$ is a closed normal subgroup of $G$ and $A_i/A_{i+1}$ is a compactly generated $FC$-group.

We note that by [2, Corollary 3.9], every compactly generated $FC$-group has a compact normal subgroup such that the corresponding factor group is of the form $V \times Z^k$. 
THEOREM 3. If $G$ is a locally compact generalized FC-group, then $G$ has a maximal compact normal subgroup.

PROOF. We proceed by induction, noting that, for $n = 1$, the conclusion follows from the corollary referred to above. Let $G = A_1 \supset A_2 \supset \cdots \supset A_{n+1} = e$, where each $A_i/A_{i+1}$ is a compactly generated FC-group. We assume that every generalized FC-group with sequence $\{A_i\}$ of shorter length has a maximal compact normal subgroup. Thus $G/A_n$ has a maximal compact normal subgroup $M$ and $A_n$ has a maximal compact normal subgroup $K$. Let $N = \pi^{-1}(M)$, where $\pi$ is the canonical mapping of $G$ onto $G/A_n$. Again, using [2, Corollary 3.9], we let $Q = Z(A_n/K, N/K)$ and note that $Q$ is a compactly generated central group. Thus $Q$ has a maximal compact normal subgroup. We let $P$ be the inverse image of this subgroup under the canonical mapping of $N$ onto $N/K$. We show that $P$ is the maximal compact normal subgroup of $G$. Let $L$ be any compact normal subgroup of $G$. Then $\pi(L) \subset M$ and $L \subset N$. Also $L/K \cap A_n/K = \{e\}$ since $A_n/K \cong V \times Z^k$. If follows that $L/K \subset Q$ and $L \subset P$ as desired.

DEFINITION 4. A topological group $G$ is an $N$-group if for every neighborhood $U$ of the identity and compact set $C$ of $G$, there is a neighborhood $V$ of the identity such that $gVg^{-1} \subset U \cup (G-C)$ for all $g \in G$. A group $G$ is a SIN-group if $G$ has small neighborhoods of the identity which are invariant under the inner automorphisms of $G$.

It is easy to see that, if a topological group $G$ can be embedded in an SIN-group, then $G$ is an $N$-group.

EXAMPLE 3. We construct an example of an $N$-group which is residual Lie and which cannot be embedded in a pro-Lie group. We begin with the group $F$ of $2 \times 2$ matrices of determinant equal to 1. Let $H$ be the subgroup of $F$ of matrices of the form \( \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \). If $N_0$ is a normal subgroup of $F \times F$ which contains all elements of the form $[A, A^{-1}]$, then $N_0$ contains all pairs of the form $[I, C]$, where $I$ is the identity and $C$ is an element of $H$. To show this, let $A = \begin{pmatrix} a/y & 1/y/a \\ 0 & y \end{pmatrix}$ and $B = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$. Then

\[
[AB, A^{-1}B^{-1}] = \begin{pmatrix} 1/y & a/y \\ 0 & 1/y \end{pmatrix}, \begin{pmatrix} y^{-1/a} & 0 \\ 0 & 1 \end{pmatrix}
\]

is in $N_0$. Thus, if

\[
A = \begin{pmatrix} a^2 - 1/ca & a \\ 0 & ca/a^2 - 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} ca/a^2 - 1 & -1/a \\ 0 & a^2 - 1/ca \end{pmatrix},
\]

then $[A, B]$ is in $N_0$. It follows that $[I, (1/c)]$ is in $N_0$ as desired, since $[I, (1/c)] = [A, B][A^{-1}, A]$.

If $c > 0$ and $C = (1/c)$, then $[I, C]$ is the conjugate of $[I, (1/c)]$ by $[A^{-1}, A]$, where

\[
A = \begin{pmatrix} \sqrt{c} & 1 \\ 0 & 1/\sqrt{c} \end{pmatrix}.
\]

If $c < 0$, then $[I, C]$ is the conjugate of $[I, (1/c)]$ by $[A^{-1}, A]$, where

\[
A = \begin{pmatrix} \sqrt{-c} & -1 \\ 0 & 1/\sqrt{-c} \end{pmatrix}.
\]

Thus, $N_1 = I \times H$ has only three conjugacy classes relative to $N_0$. We show that $N_0$ cannot be embedded (isomorphically) in a compact group.
LEMMA. If $G$ is a discrete group and $H$ is an infinite subgroup which has only finitely many conjugacy classes relative to $G$, then $G$ cannot be embedded in a compact group.

PROOF. Suppose there exists an isomorphism $Q: G \rightarrow K$, a compact group. If $H_0$ is a conjugacy class of $H$ such that $H_0 \neq \{e\}$, then there is a neighborhood $V$ of the identity of $K$ such that $Q(H_0) \cap V = \emptyset$. Because, if $Q(H_0)$ intersects every neighborhood of the identity of $K$, then for any neighborhood $W$ there is a neighborhood $V$ of the identity of $K$ such that $gVg^{-1} \subset W$. If $Q(H_0) \cap V \neq \emptyset$, then $Q(H_0) \subset W$. Consequently $Q(H_0) = e$ which is a contradiction. Since $H$ has only finitely many conjugacy classes, there is a neighborhood $U$ of the identity of $K$ such that $Q(H) \cap U = e$. Thus $Q(H)$ is discrete which contradicts the fact that $K$ is compact.

We are ready to construct the example referred to above. Let $F_i = F \times F$, where $F$ is the group of $2 \times 2$ matrices referred to above and $K_i = Z_2$ for each positive integer $i$. Let $G = \sum F_i \times_{\eta} \prod K_i$, where $\sum F_i$ is the direct sum with discrete topology, $\prod K_i$ is the direct product with product topology, and $\eta$ is defined as follows: If the $i$th coordinate of $k \in \prod K_i$ is 1, then the $i$th coordinate of $\eta(k)(f)$ is $(f_2, f_1)$, where the $i$th coordinate of $f$ is $(f_1, f_2)$. If the $i$th coordinate of $k$ is 0, then, of course, $\eta(k)$ does not alter the $i$th coordinate of $f \in \sum F_i$.

We note that any open normal subgroup of $G$ contains an isomorphic image of a subgroup $N_0$ of $F \times F$ such that $[A, A^{-1}] \in N_0$ for each $A \in F$. To see this we note that, if $k \in \prod K_i$ has $i$th coordinate 1 and all other coordinates 0, and $f \in \sum F_i$ has $i$th coordinate $[A, I]$, and all other coordinates $[I, I]$, then $(f, 0) \in \sum F_i \times_{\eta} \prod K_i$ and $(f, 0)(I, k)(f, 0)^{-1} = (f', k)$, where $f'$ has $i$th coordinate $[A, A^{-1}]$ and all other coordinates $[I, I]$, where $I$ here is the identity of $F$.

To show that $G$ cannot be embedded in a pro-Lie group, assume there exists a continuous isomorphism $Q: F \rightarrow G_1$ and $G_1$ is pro-Lie. Let $K$ be a normal compact subgroup of $G_1$ such that $G_1/K$ is a Lie group. Then $G/Q^{-1}(K)$ is a Lie group, hence discrete since $G$ is totally disconnected. It follows that $Q^{-1}(K)$ is an open normal subgroup of $G$. This contradicts the Lemma. Thus, the existence of the continuous isomorphism $Q$ is contradicted.

Since a locally compact SIN-group is pro-Lie [2, Theorem 2.11], it follows from the above argument that $G$ cannot be embedded in a locally compact SIN-group.

In conclusion we see that $G$ is an $N$-group and residual Lie. Actually $G$ can be embedded in $\sum F_i \times_{\eta} \prod K_i$, where $\sum F_i \times \prod K_i$ has the relative product topology of $\prod F_i \times \prod K_i$ which is SIN but, of course, not locally compact.

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