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CANONICAL QUOTIENT SINGULARITIES IN DIMENSION THREE

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ABSTRACT. We classify isolated canonical cyclic quotient singularities in dimension three, showing that, with two exceptions, they are all either Gorenstein or terminal. The proof uses the solution of a combinatorial problem which arose in the study of algebraic cycles on Fermat varieties.

This note is a continuation of our previous paper **[7]** (written jointly with Glenn Stevens), which studied certain canonical quotient singularities in dimensions three and four. The classification of canonical quotient singularities can be reduced to a certain combinatorial problem, which we partially solved in **[7]**. After **[7]** went to press, we learned that a similar (although not quite identical) combinatorial problem occurs in the study of algebraic cycles on Fermat varieties (cf. **[6, 8, 10]**). In this note, we apply the recent solution of that problem (due to Aoki and Shioda **[1, 2, 10]**) to the classification of canonical quotient singularities in dimension three.

If x is a rational number, we let $\langle x \rangle$ denote the rational number such that $x \equiv \langle x \rangle \bmod \mathbf{Z}$, and $0 \leqslant \langle x \rangle < 1$.

DEFINITION. A *Fermat quadruple* is a quadruple of rational numbers $(a_1/N, a_2/N, a_3/N, a_4/N)$ such that $a_i \not\equiv 0 \pmod N$, g.c.d.$\{a_i\} = 1$, and, for every integer k with $(k, N) = 1$, $\Sigma_i \langle a_i k/N \rangle = 2$.

The following theorem was conjectured by Meyer and Neutsch **[6]** and Shioda **[10]**, and proved by Aoki and Shioda **[1, 2, 10]**.

THEOREM 1. *Let $(a_1/N, a_2/N, a_3/N, a_4/N)$ be a Fermat quadruple. Then there is some k with $(k, N) = 1$ such that $(\langle a_1k/N \rangle, \langle a_2k/N \rangle, \langle a_3k/N \rangle, \langle a_4k/N \rangle)$ is, after reordering, one of the following*:

(i) $(\alpha/N, 1 - (\alpha/N), \beta/N, 1 - (\beta/N))$ *for some* $0 < \alpha, \beta < N$,

(ii) $(1/2K, K/2K, (K + 1)/2K, (2K - 2)/2K)$ *where* $N = 2K$,

(iii) $(1/2K, (K + 1)/2K, (K + 2)/2K, (2K - 4)/2K)$ *where* $N = 2K$,

(iv) $(1/3K, (K + 1)/3K, (2K + 1)/3K, (3K - 3)/3K)$ *where* $N = 3K$,

(v) *one of* 101 *"exceptional quadruples" listed in* **[6]**: *each has* $(N, 6) \neq 1$ *and* $N \leqslant 180$.

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The combinatorial fact which we used in [**7**] to classify terminal quotient singularities is

THEOREM 2 [**13**, **3**, **4** AND **7**]. *Let a, b, c be integrers relatively prime to N. Suppose, for all $k \in \mathbf{Z} - N\mathbf{Z}$,*

$$\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle > 1.$$

Then, after reordering a, b, c, we have $a + b \equiv 0 \pmod N$.

Note that under the hypotheses of Theorem 2, if we let $d = -(a + b + c)$, then $(a/N, b/N, c/N, d/N)$ is a Fermat quadruple. For N prime, Theorems 1 and 2 are equivalent; this case of these theorems was also proved by W. Parry (cf. [**6** and **8**]).

We can now state our main combinatorial result.

THEOREM 3. *Let a, b, c be integers relatively prime to N. Suppose, for all $k \in \mathbf{Z} - N\mathbf{Z}$,*

$$(*) \qquad \langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle \geqslant 1.$$

Then either

(i) $\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle \in \mathbf{Z}$ *for all k,*

(ii) *after reordering a, b, c, we have $a + b \equiv 0 \pmod N$, or*

(iii) *there is some integer k with $(k, N) = 1$ such that, after reordering,*

$$(\langle ak/N \rangle, \langle bk/N \rangle, \langle ck/N \rangle) = (1/9, 4/9, 7/9) \text{ or } (1/14, 9/14, 11/14).$$

Conversely, if either (i), (ii), *or* (iii) *holds, then* $(*)$ *is true for all $k \in \mathbf{Z} - N\mathbf{Z}$.*

PROOF. Let $d = -(a + b + c)$. If $d \equiv 0 \pmod N$ then we have case (i). If $(d, N) = 1$ then, for each $k \in \mathbf{Z} - N\mathbf{Z}$, $\langle dk/N \rangle > 0$, so that

$$\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle + \langle dk/N \rangle \geqslant 1 + \langle dk/N \rangle > 1.$$

We may then apply Theorem 2 to conclude that we have case (ii).

Thus, we may assume that $d \not\equiv 0 \pmod N$ and $(d, N) \neq 1$. In this case, $(a/N, b/N, c/N, d/N)$ is a Fermat quadruple in which exactly three of the numerators are relatively prime to N. We apply Theorem 1 to find such Fermat quadruples. In case (i) of Theorem 1, either 0, 2, or 4 of the numerators are relatively prime to N, and, in cases (ii) and (iii), at most two of the numerators are relatively prime to N. Thus, any Fermat quadruples satisfying our hypothesis will be found in cases (iv) and (v).

Suppose $(1/3K, (K + 1)/3K, (2K + 1)/3K, (3K - 3)/3K)$ is a Fermat quadruple (with $N = 3K$) from case (iv) of Theorem 1. In order to satisfy our hypothesis, 1, $K + 1$, and $2K + 1$ must all be relatively prime to $3K$; in particular, $K \equiv 0 \pmod 3$. On the other hand,

$$\langle 3/3K \rangle + \langle 3(K + 1)/3K \rangle + \langle 3(2K + 1)/3K \rangle = 3/K \geqslant 1$$

so that $K \leqslant 3$. We thus find that the only solution in this case is $K = 3$ and $(\langle ak/N \rangle, \langle bk/N \rangle, \langle ck/N \rangle) = (1/9, 4/9, 7/9)$.

To analyse case (v) of Theorem 1, we have listed in Table 1 all exceptional Fermat quadruples in which exactly three of the numerators are relatively prime to N. We have also (in all but the first line of the table), listed a value of k and the corresponding triple $(\langle ak/N\rangle, \langle bk/N\rangle, \langle ck/N\rangle)$; since the sum in each case is less than 1, none of these satisfy our hypothesis. On the other hand, it is easily verified that the remaining case $(1/14, 9/14, 11/14)$ satisfies the hypothesis for each k. Q.E.D.

$(a/N, b/N, c/N, d/N)$	k	$(\langle ak/N\rangle, \langle bk/N\rangle, \langle ck/N\rangle)$
$(1/14, 9/14, 11/14, 7/14)$	—	——
$(1/21, 4/21, 19/21, 18/21)$	6	$(2/7, 1/7, 3/7)$
$(1/28, 9/28, 25/28, 21/28)$	7	$(1/4, 1/4, 1/4)$
$(1/42, 31/42, 37/42, 15/42)$	7	$(1/6, 1/6, 1/6)$
$(1/42, 25/42, 37/42, 21/42)$	7	$(1/6, 1/6, 1/6)$
$(1/78, 55/78, 61/78, 39/78)$	13	$(1/6, 1/6, 1/6)$

TABLE 1

From Theorem 3, we deduce a classification of isolated canonical cyclic quotient singularities in dimension three. Recall that a cyclic group $G \subset \mathrm{Gl}(3, \mathbf{C})$ of order N gives an isolated singularity X/G exactly when all eigenvalues of a generator g are primitive Nth roots of unity.

THEOREM 4. *Let $G \subset \mathrm{Gl}(3, \mathbf{C})$ be a cyclic group of order N such that all eigenvalues of a generator g are primitive Nth roots of unity. Then the isolated singularity X/G is canonical if and only if one of the following holds*:

(i) *X/G is Gorenstein,*

(ii) *X/G is terminal,*

(iii) *G is conjugate in* $\mathrm{Gl}(3, \mathbf{C})$ *to the group generated by*

$$\mathrm{diag}(e^{2\pi i/9}, e^{8\pi i/9}, e^{14\pi i/9}),$$

or to that generated by

$$\mathrm{diag}(e^{\pi i/7}, e^{9\pi i/7}, e^{11\pi i/7}).$$

PROOF. Let $\zeta = e^{2\pi i/N}$, and choose coordinates so that $g = \mathrm{diag}(\zeta^a, \zeta^b, \zeta^c)$, with $(a, N) = (b, N) = (c, N) = 1$. By the criterion of Reid [**9**], Shepherd-Barron, and Tai [**11**], X/G is canonical if and only if

$$\langle ak/N\rangle + \langle bk/N\rangle + \langle ck/N\rangle \geqslant 1$$

for all $k \in \mathbf{Z} - N\mathbf{Z}$. On the other hand, by the theorem of Khinich [**5**] and Watanabe [**12**], X/G is Gorenstein if and only if $\det g = 1$ (which holds if and only if $\langle ak/N\rangle + \langle bk/N\rangle + \langle ck/N\rangle \in \mathbf{Z}$ for all k). Since we proved in [**7**] that X/G is terminal if and only if case (ii) in Theorem 3 holds, this theorem now follows immediately from Theorem 3. Q.E.D.

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