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ON SCOTT MODULES AND p-PERMUTATION MODULES: AN APPROACH THROUGH THE BRAUER MORPHISM

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ABSTRACT. Following Lluis Puig we give a presentation of the theory of p-permutation modules (also called "trivial source modules") by a systematic use of the generalized Brauer morphism.

The aim of this paper is to present a somewhat new and self-contained treatment of p-permutation modules and Scott modules by a systematic use of the Brauer morphism, as suggested by L. Puig (private communication).

By "self-contained" we mean that only knowledge of basic facts of representation theory is needed: elementary theory of vertices and sources as presented in [**5**] (see also [**4**]), as well as classical results about lifting idempotents. Burry's result about the module-theoretic interpretation of the coefficients of lower defect groups is obtained as a by-product of that presentation.

Let G be a finite group, and let $\mathcal{O}$ be a commutative ring, complete for a discrete valuation, with maximal ideal $\mathfrak{p}$ and residual field $\mathbf{F} = \mathcal{O}/\mathfrak{p}$. We assume $\mathbf{F}$ has characteristic $p > 0$. Note that $\mathcal{O}$ may be equal to $\mathbf{F}$.

(0.1) DEFINITION. Let M be an $\mathcal{O}$-free $\mathcal{O}G$-module. We say that M is a p-permutation module if, whenever P is a p-subgroup of G, there is an $\mathcal{O}$-basis of M which is stabilized by P.

(0.2) PROPOSITION. (1) *If M and M' are two p-permutation $\mathcal{O}G$-modules, so are the modules $M \oplus M'$ and $M \otimes M'$.*

(2) *Let H be a subgroup of G. If M (resp. N) is a p-permutation $\mathcal{O}G$-module (resp. $\mathcal{O}H$-module), then $\mathrm{Res}_H^G(M)$ (resp. $\mathrm{Ind}_H^G(N)$) is a p-permutation $\mathcal{O}H$-module (resp. $\mathcal{O}G$-module).*

(3) *Any summand of a p-permutation module is a p-permutation module.*

The first two assertions are obvious. Let us prove the third. Let M be a p-permutation $\mathcal{O}G$-module, and let M' be a summand of M. If P is a p-subgroup of G, then $\mathrm{Res}_P^G(M')$ is a summand of $\mathrm{Res}_P^G(M)$. By definition $\mathrm{Res}_P^G(M)$ is a direct sum of modules isomorphic to $\mathrm{Ind}_Q^P(1_{\mathcal{O}Q})$ (Q subgroups of P). Thus the assertion will result from the following lemma.

(0.3) LEMMA. *Let P be a p-group, let Q be a subgroup of P. Then the $\mathcal{O}G$-module $\mathrm{Ind}_Q^P(1_{\mathcal{O}Q})$ is indecomposable.*

That result is well known and a consequence of Green's theorem about induced modules (see e.g. [**4**, Theorem 3.8]). We present here an elementary proof due to M. Cabanes. It suffices to check that $\mathrm{Ind}_Q^P(1_{\mathbf{F}Q})$ is indecomposable. Since a p-group always has nontrivial fixed points on a nontrivial $\mathbf{F}$-vector space, it suffices

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to prove that $(\mathrm{Ind}_Q^P(1_{\mathbf{F}Q}))^P$ has dimension 1 over $\mathbf{F}$; that last property results from Frobenius reciprocity, since

$$(\mathrm{Ind}_Q^P(1_{\mathbf{F}Q}))^P = \mathrm{Hom}_{\mathbf{F}P}(1_{\mathbf{F}P}, \mathrm{Ind}_Q^P(1_{\mathbf{F}Q})) = \mathrm{Hom}_{\mathbf{F}Q}(1_{\mathbf{F}Q}, 1_{\mathbf{F}Q}) = \mathbf{F}.$$

The following characterization shows that p-permutation modules are in fact familiar objects.

(0.4) *Let M be an indecomposable $\mathcal{O}G$-module. The module M is a p-permutation $\mathcal{O}G$-module if and only if one of the following holds*:

(i) *there exists a subgroup H of G such that M is isomorphic to a summand of* $\mathrm{Ind}_H^G(1_{\mathcal{O}H})$;

(ii) *M has trivial source.*

By (0.2)(3) any summand of $\mathrm{Ind}_H^G(1_{\mathcal{O}H})$ (hence, any module with trivial source) is a p-permutation module. Conversely, suppose that M is a p-permutation module, and let P be a p-subgroup such that M is P-projective; then M is a summand of $\mathrm{Ind}_P^G\mathrm{Res}_P^G(M)$. But by definition $\mathrm{Res}_P^G(M)$ is a direct sum of modules $\mathrm{Ind}_Q^P(1_{\mathcal{O}Q})$ (Q subgroups of P); thus there exists a subgroup Q of P such that M is a summand of $\mathrm{Ind}_Q^P(1_{\mathcal{O}Q})$. Moreover, if P is a vertex of M, necessarily $Q = P$, proving that M has a trivial source.

1. On the Brauer morphism. We first need to recall some definitions (see [**1, 2, 5**]). Whenever M is an $\mathcal{O}G$-module and H and H' are two subgroups of G such that $H \subset H'$, we denote by $\mathrm{Tr}_H^{H'}$ the linear map from M^H (the set of fixed points of H in M) into $M^{H'}$ defined by (see [**5**]) $\mathrm{Tr}_H^{H'}(x) = \sum g(x)$, where g runs over a complete set of representatives in H' of H'/H. We set $M_H^{H'} = \mathrm{Tr}_H^{H'}(M^H)$.

Whenever P is a p-subgroup of G, we put $\overline{N}_G(P) = N_G(P)/P$ and denote by $M(P)$ the $\mathbf{F}\overline{N}_G(P)$-module defined by (see [**1** or **2**])

$$M(P) = M^P \Big/ \left(\sum_Q M_Q^P + \mathfrak{p}M^P \right)$$

where Q runs over the set of proper subgroups of P. We call "Brauer morphism" the natural surjection $\mathrm{Br}_P^M : M^P \to M(P)$. The Brauer morphism is a morphism of $\mathcal{O}\overline{N}_G(P)$-modules.

We shall use the following easy results (see e.g. [**1** or **2**]).

(1.1) (1) *Suppose A is an $\mathcal{O}G$-algebra. Then* $\mathrm{Ker}(\mathrm{Br}_P^A)$ *is a two-sided ideal of A^P, $A(P)$ is an $\mathbf{F}\overline{N}_G(P)$-algebra, and* Br_P^A *is a morphism of $\mathcal{O}\overline{N}_G(P)$-algebras.*

(2) *We have*

$$\mathrm{Tr}_1^{\overline{N}_G(P)} \circ \mathrm{Br}_P^M = \mathrm{Br}_P^M \circ \mathrm{Tr}_P^G,$$

and, in particular,

$$\mathrm{Br}_P^M(M_P^G) = (M(P))_1^{\overline{N}_G(P)}.$$

(3) *Suppose* $\mathrm{Res}_P^G(M)$ *is a permutation $\mathcal{O}P$-module with a basis X stabilized by P. Then $M(P)$ has for $\mathbf{F}$-basis the set* $\mathrm{Br}_P^M(C_X(P))$ *in bijection with the set $C_X(P)$ of fixed points of X under P. Moreover, the $\mathbf{F}\overline{N}_G(P)$-modules $M(P)$ and $\overline{M}(P)$ (where $\overline{M} = M/\mathfrak{p}M$) are canonically isomorphic.*

We also need some other properties.

(1.2) *Let M_1, M_2, and M be $\mathcal{O}G$-modules, and suppose that $f\colon M_1 \times M_2 \to M$ is a bilinear map stable under G-action. Then f induces a bilinar map $f_P\colon M_1(P) \times M_2(P) \to M(P)$ stable under $\overline{N}_G(P)$-action such that*

$$f_P(\mathrm{Br}_P^{M_1}(x_1), \mathrm{Br}_P^{M_2}(x_2)) = \mathrm{Br}_P^M(f(x_1, x_2))$$

for all x_1 in M_1 and x_2 in M_2.

The proof of (1.2) is straightforward and left to the reader.

(1.3) *Let H be a subgroup of G and let P be a p-group of G. Let M be an H-projective $\mathcal{O}G$-module. Then $M(P) = 0$ unless P is G-conjugate to a subgroup of H. In particular, if N is any $\mathcal{O}H$-module, then $(\mathrm{Ind}_H^G(N))(P) = 0$ unless P is G-conjugate to a subgroup of H.*

Indeed, let $A = \mathrm{End}_{\mathcal{O}}(M)$. By Higman's criteria, $\mathrm{id}_M \in A_H^G$, but (see [**1** or **5**]) $A_H^G \subset \sum_{g\in G} A_{P\cap gHg^{-1}}^P$, from which it follows that $M^P \subset \sum_{g\in G} M_{P\cap gHg^{-1}}^P$, which establishes the assertion.

(1.4) *Let H be a subgroup of G, P a p-subgroup of G, and $T_G(P,H)$ the set of all g in G such that $P^g \subset H$. Then as $\mathbf{F}\overline{N}_G(P)$-modules, we have an isomorphism*

$$(\mathrm{Ind}_H^G(1_{\mathcal{O}H}))(P) \simeq \sum_g \mathrm{Ind}_{N_{gHg^{-1}}(P)}^{N_G(P)}(1_{\mathbf{F}N_{gHg^{-1}}(P)}),$$

where g runs over a set of representatives in $T_G(P,H)$ of $N_G(P)\backslash T_G(P,H)/H$.

Indeed, by Mackey's theorem, we have

$$\mathrm{Res}_{N_G(P)}^G \mathrm{Ind}_H^G(1_{\mathcal{O}H}) \simeq \sum_g \mathrm{Ind}_{N_G(P)\cap {}^gH}^{N_G(P)}(1_{\mathcal{O}(N_G(P)\cap {}^gH)}),$$

where g runs over a complete set of representatives of $N_G(P)\backslash G/H$. Now (1.4) follows from the fact that (by (1.3))

$$(\mathrm{Ind}_{N_G(P)\cap {}^gH}^{N_G(P)}(1_{\mathcal{O}(N_G(P)\cap {}^gH)}))(P) = 0 \quad \text{if } P \not\subset {}^gH,$$

and if $P \subset {}^gH$, then P acts trivially on $\mathrm{Ind}_{N_{{}^gH}(P)}^{N_G(P)}(1_{\mathcal{O}N_{{}^gH}(P)})$.

2. Scott modules and Scott coefficients associated with a p-subgroup. Alperin and Scott proved the following result.

(2.1) *Let P be a p-subgroup of G. There exists an indecomposable p-permutation $\mathbf{F}G$-module with vertex P denoted by $S_P(G,\mathbf{F})$, uniquely determined up to isomorphism by one of the following properties:*

(i) *$1_{\mathbf{F}G}$ is isomorphic to a submodule of $S_P(G,\mathbf{F})$;*

(ii) *$1_{\mathbf{F}G}$ is isomorphic to a quotient of $S_P(G,\mathbf{F})$.*

Moreover, the module $S_P(G,\mathbf{F})$ is isomorphic to its dual and is a summand of $\mathrm{Ind}_H^G(1_{\mathbf{F}H})$ if and only if P is G-conjugate to a Sylow p-subgroup of H.

The module $S_P(G,\mathbf{F})$ is called the *Scott module of G associated to P.*

We obtain a new proof of that result as a by-product of the definition and the study (due to Lluis Puig and suggested to him by some ideas of J. A. Green in [**6**]), every p permutation $\mathcal{O}G$-module M, of an integer, denoted by $s_P(M)$ and called the "Scott coefficient of M associated to P", which will be shown to have the following property: the integer $s_P(M)$ is the number of factors isomorphic to $S_P(G,\mathbf{F})$ in a decomposition of $\overline{M}$ into a direct sum of indecomposable modules.

(2.2) DEFINITION (L. PUIG). *Let M be a p-permutation $\mathcal{O}G$-module, and let P be a p-subgroup of G. The Scott coefficient of M associated with P is*

$$s_P(M) = \dim_{\mathbf{F}}(\mathrm{Br}_P^M(M_P^G)).$$

By (1.1)(2) we see that

(2.3) *We have*

$$s_P(M) = \dim_{\mathbf{F}}((M(P))_1^{\overline{N}_G(P)}).$$

The main results about Scott coefficients and Scott modules are, as we shall see, easy consequences of the following lemma.

(2.4) LEMMA. *Let P be a p-subgroup of G.*

(1) *If M and M' are two p-permutation $\mathcal{O}G$-modules, $s_P(M \oplus M') = s_P(M) + s_P(M')$.*

(2) *Let M be a p-permutation $\mathcal{O}G$-module and let $M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ be its dual. Then $s_P(M) = s_P(M^*)$.*

(3) *Let H be a subgroup of G. Then $s_P(\mathrm{Ind}_H^G(1_{\mathcal{O}H})) = 0$ unless P is G-conjugate to a Sylow p-subgroup of H, in which case $s_P(\mathrm{Ind}_H^G(1_{\mathcal{O}H})) = 1$.*

The first assertion is obvious. We prove the second. Let X be an $\mathcal{O}$-basis of M stable under P. Then the dual basis X^* is an $\mathcal{O}$-basis of M^* stable by P, and the operations of P on X and on X^* are isomorphic. By (1.2) the canonical duality $M \times M^* \to \mathcal{O}$ induces a $\overline{N}_G(P)$-duality $M(P) \times M^*(P) \to \mathbf{F}$; hence, it induces an $\mathbf{F}\overline{N}_G(P)$-homomorphism $M^*(P) \to M(P)^*$ which obviously sends the basis $\mathrm{Br}_P^{M^*}(C_{X^*}(P))$ onto the dual basis of $\mathrm{Br}_P^M(C_X(P))$; hence, it is an isomorphism. Thus, $s_P(M^*)$ is equal to the rank of the endomorphism $\mathrm{Tr}_1^{\overline{N}_G(P)}$ of $M(P)^*$, which is equal to the rank of its transpose, i.e. the endomorphism $\mathrm{Tr}_1^{\overline{N}_G(P)}$ of $M(P)$; hence, $s_P(M) = s_P(M^*)$.

Let us now prove the third assertion. According to property (1.3), we see that $(\mathrm{Ind}_H^G(1_{\mathcal{O}H}))(P) = 0$ if P is not conjugate to a subgroup of H. By Sylow theorems in H, we know that if P^g is a Sylow p-subgroup in H for some g in G, then $P^{g'}$ is also a Sylow p-subgroup of H only for those g' lying in $N_G(P)gH$. So by (1.4), we see that it suffices to prove that $(\mathrm{Ind}_H^G(1_{\mathbf{F}H}))_1^G$ has dimension 0 or 1 according to the fact that H is, or is not, a p'-group: this is easy to check.

Now we can prove, following Puig's method, the main result about Scott modules.

(2.5) THEOREM (SCOTT, ALPERIN). *Let P be a p-subgroup of G.*

(1) *There exists a unique indecomposable p-permutation $\mathcal{O}G$-module $S_P(G, \mathcal{O})$ such that $s_P(S_P(G, \mathcal{O})) \neq 0$. We have $s_P(S_P(G, \mathcal{O})) = 1$, and $S_P(G, \mathcal{O})$ is isomorphic to its dual.*

(2) *If H is a subgroup of G, then $S_P(G, \mathcal{O})$ is isomorphic to a summand of $\mathrm{Ind}_H^G(1_{\mathcal{O}H})$ if and only if P is G-conjugate to a Sylow p-subgroup of H. In this case $S_P(G, \mathcal{O})$ is the unique indecomposable summand M of $\mathrm{Ind}_H^G(1_{\mathcal{O}H})$ such that one of the following holds:*

(i) $\mathrm{Hom}_{\mathcal{O}G}(1_{\mathcal{O}H}, M) \neq 0$,

(ii) $\mathrm{Hom}_{\mathcal{O}G}(M, 1_{\mathcal{O}G}) \neq 0$,

and we have $\mathrm{Hom}_{\mathcal{O}G}(1_{\mathcal{O}G}, M) \simeq \mathrm{Hom}_{\mathcal{O}G}(M, 1_{\mathcal{O}G}) \simeq \mathcal{O}$.

PROOF OF (2.5)(1). Any p-permutation $\mathcal{O}G$-module M is a summand of some $\mathrm{Ind}_Q^G(1_{\mathcal{O}Q})$ for a p-subgroup Q of G (see (0.4)). If $s_P(M) \neq 0$, by (2.4)(1), (3), we see that P is G-conjugate to Q, and since $s_P(\mathrm{Ind}_P^G(1_{\mathcal{O}P})) = 1$, we see that M is the unique indecomposable summand of $\mathrm{Ind}_P^G(1_{\mathcal{O}P})$ such that $s_P(M) \neq 0$; we have $s_P(M) = 1$, and by the unicity of M it follows from (2.4)(2) that $M \simeq M^*$.

PROOF OF (2.5)(2). By (1) we see that $S_P(G, \mathcal{O})$ is a summand of $\mathrm{Ind}_H^G(1_{\mathcal{O}H})$ if and only if $s_P(\mathrm{Ind}_H^G(1_{\mathcal{O}H})) \neq 0$, so by (2.4)(3) if P is G-conjugate to a Sylow p-subgroup of H. Moreover, it follows from Frobenius reciprocity that

$$\mathrm{Hom}_{\mathcal{O}G}(1_{\mathcal{O}G}, \mathrm{Ind}_P^G(1_{\mathcal{O}P})) \simeq \mathcal{O} \simeq \mathrm{Hom}_{\mathcal{O}G}(\mathrm{Ind}_P^G(1_{\mathcal{O}P}), 1_{\mathcal{O}G}).$$

Thus, in order to complete the proof of (2.5) and since $S_P(G, \mathcal{O}) \simeq S_P(G, \mathcal{O})^*$, it suffices to prove that $\mathrm{Hom}_{\mathcal{O}G}(1_{\mathcal{O}G}, S_P(G, \mathcal{O})) \neq 0$. But by (1.1)(2) we know that $(S_P(G, \mathcal{O}))_P^G$ is mapped *onto* $((S_P(G, \mathcal{O}))(P))_1^{\overline{N}_G(P)}$ by the Brauer morphism, and that last module is not zero by definition of $S_P(G, \mathcal{O})$ since its dimension is precisely $s_P(S_P(G, \mathcal{O}))$; so, in particular, $S_P(G, \mathcal{O})^G \neq 0$.

Let us notice now that Burry's nice result [**3**] is an immediate consequence of that presentation.

(2.6) COROLLARY (BURRY). *Let e be an idempotent of $Z\mathbf{F}G$, and let P be a p-subgroup of G. Then the Brauer coefficient of P associated with e is the multiplicity of $S_P(G, \mathbf{F})$ as a summand of $(\mathbf{F}G)e$, where G acts by conjugation.*

Indeed, the coefficient $m_e(P)$ of P associated with e is (see [**1**, §II.1])

$$m_e(P) = \dim_{\mathbf{F}}(\mathrm{Br}_P((\mathbf{F}G)_P^G)),$$

i.e., precisely by Definition (2.2)

$$m_e(P) = s_P((\mathbf{F}G)e).$$

(2.7) REMARK. Let M be any p-permutation G-module, and let $\overline{M} = M/\mathfrak{p}M$. By (1.1)(3) we see that $s_P(M) = s_P(\overline{M})$. From the characterization of $S_P(G, \mathcal{O})$ given in (2.5) it follows then that $\overline{S_P(G, \mathcal{O})} = S_P(G, \mathbf{F})$.

3. p-permutation modules through the Brauer morphism. As we shall see, the Brauer morphism is particularly convenient for the local study of p-permutation $\mathcal{O}G$-modules.

(3.1) *Let M be a p-permutation $\mathcal{O}G$-module, and let P be a p-subgroup of G. Then $M(P)$ is a p-permutation $\mathbf{F}\overline{N}_G(P)$-module.*

Indeed, let Q be a Sylow p-subgroup of $N_G(P)$, and let X be a Q-stable $\mathcal{O}$-basis of M. Then (see (1.1)(3)) $\mathrm{Br}_P^M(C_X(P))$ is a (Q/P)-stable $\mathbf{F}$-basis of $M(P)$.

The next statement is an omnibus theorem giving the main properties of the Brauer morphism applied to p-permutation modules.

(3.2) THEOREM. (1) *The vertices of an indecomposable p-permutation $\mathcal{O}G$-module M are the maximal p-subgroup P of G such that $M(P) \neq 0$.*

(2) *An indecomposable p-permutation $\mathcal{O}G$-module M has vertex P if and only if $M(P)$ is nontrivial and a projective $\mathbf{F}\overline{N}_G(P)$-module.*

(3) *The correspondence $M \to M(P)$ induces a bijection between the isomorphism classes of indecomposable p-permutation $\mathcal{O}G$-modules with vertex P and the*

isomorphism classes of indecomposable projective $\mathbf{F}\overline{N}_G(P)$*-modules. In particular,* $(S_P(G,\mathcal{O}))(P)$ *is the projective cover of* $1_{\mathbf{F}\overline{N}_G(P)}$.

(4) *Let* M *be a p-permutation* $\mathcal{O}G$*-module, let* E *be an indecomposable projective* $\mathbf{F}\overline{N}_G(P)$*-module, and let* $M(P,E)$ *be the corresponding p-permutation* $\mathcal{O}G$*-module with vertex* P*. Then* $M(P,E)$ *is a summand of* M *if and only if* E *is a summand of* $M(P)$.

Part of the proof of (3.2) may be simplified by using the Green correspondence (see (3.4)). But in order for the presentation to be "self-contained", we give an independent proof.

By (1.3) we see that it suffices to prove that $M(P)\neq 0$ whenever P is a vertex of M. Suppose M is a summand of $\mathrm{Ind}_P^G(1_{\mathcal{O}P})$. Then by Mackey's theorem and by (0.3), $\mathrm{Res}_P^G(M)$ is isomorphic to a direct sum of modules of type $\mathrm{Ind}_{P\cap {}^gP}^P(1_{\mathcal{O}(P\cap {}^gP)})$, where g runs over a certain subset of G. But M is a summand of $\mathrm{Ind}_P^G\mathrm{Res}_P^G(M)$, and since P is a vertex of M, we see that in the set of g's there is at least one element in $N_G(P)$. In other words, $1_{\mathcal{O}P}$ is a summand of $\mathrm{Res}_P^G(M)$, proving that $M(P)\neq 0$.

To prove (2)–(4) of (3.2), we need

(3.3) LEMMA (PUIG). *Let* P *be a p-subgroup of* G*, let* M *be a p-permutation* $\mathcal{O}G$*-module, and let* $A=\mathrm{End}_{\mathcal{O}}(M)$*. Then the natural operation of* $A(P)$ *over* $M(P)$ *induces an isomorphism of* $\mathbf{F}\overline{N}_G(P)$*-algebras between* $A(P)$ *and* $\mathrm{End}_{\mathbf{F}}(M(P))$*. Moreover, for* $a\in A^P$ *and* $x\in M^P$*, we have*

$$\mathrm{Br}_P^A(a)(\mathrm{Br}_P^M(x))=\mathrm{Br}_P^M(a(x)).$$

The natural bilinar map $A\times M\to M$ induces a bilinar map $A(P)\times M(P)\to M(P)$ which is stable under $\overline{N}_G(P)$ (see (1.2)), hence an $\mathbf{F}\overline{N}_G(P)$-morphism $A(P)\to\mathrm{End}_{\mathbf{F}}(M(P))$ satisfying the last condition of the lemma. Let us prove that this morphism is an isomorphism. Let X be an $\mathcal{O}$-basis of M stable under P. Then the set $X(A)=\{a_{x,y}|x,y\in X\}$, where $a_{x,y}(z)=\delta_{y,z}x$ for x,y,z in X, is a P-stable basis of A. It is clear that $C_{X(A)}(P)=\{a_{x,y}|x,y\in C_X(P)\}$. For $x\in C_X(P)$ or $a\in C_{X(A)}(P)$, let us set $\overline{x}=\mathrm{Br}_P^M(x)$ and $\overline{a}=\mathrm{Br}_P^A(a)$. Then we have $\overline{a}_{x,y}(\overline{z})=\delta_{\overline{y},\overline{z}}\overline{x}$ for x,y,z in $C_X(P)$, which proves (see (1.1)(3)) that the morphism $A(P)\to\mathrm{End}_{\mathbf{F}}(M(P))$ is an isomorphism.

PROOF OF (3.2)(2). Suppose first that M is an indecomposable p-permutation $\mathcal{O}G$-module such that $M(P)$ is a nontrivial $\mathbf{F}\overline{N}_G(P)$-projective module; let us denote $\mathrm{End}_{\mathcal{O}}(M)$ by A. Then by (3.3) and the Higman criteria,

$$A(P)^{\overline{N}_G(P)}=(A(P))_1^{\overline{N}_G(P)}.$$

But $(A(P))_1^{\overline{N}_G(P)}$ is the image, through the Brauer morphism Br_P^A, of A_P^G (see (1.1)(2)). Now by results about lifting idempotents, since id_M is the unique nonzero idempotent of A^G, we deduce that $\mathrm{id}_M\in A_P^G$, proving that M is P-projective and M has P as a vertex by (3.2)(1).

Conversely, if M has vertex P, we have $\mathrm{id}_M\in A_P^G$, so $\mathrm{id}_{M(P)}\in(A(P))_1^{\overline{N}_G(P)}$, which (with (3.3)) proves that $M(P)$ is a projective $\mathbf{F}\overline{N}_G(P)$-module.

PROOF OF (3.2)(3). As a consequence of (1.4) we have

$$(\mathrm{Ind}_P^G(1_{\mathcal{O}P}))(P)=\mathrm{Ind}_1^{\overline{N}_G(P)}(1_{\mathbf{F}}).$$

The indecomposable p-permutation $\mathcal{O}G$-modules with vertex P correspond to the summands of $\mathrm{Ind}_P^G(1_{\mathcal{O}P})$ with vertex P; the indecomposable projective $\mathbf{F}\overline{N}_G(P)$-modules correspond to summands of $\mathrm{Ind}_1^{\overline{N}_G(P)}(1_\mathbf{F})$. Setting $A = \mathrm{End}_\mathcal{O}(\mathrm{Ind}_P^G(1_{\mathcal{O}P}))$ by (3.3) we know that $A(P) \simeq \mathrm{End}_\mathbf{F}(\mathrm{Ind}_1^{\overline{N}_G(P)}(1_\mathbf{F}))$ as $\mathbf{F}\overline{N}_G(P)$-algebras. Now the summands of $\mathrm{Ind}_P^G(1_{\mathcal{O}P})$ with vertex P correspond to the primitive idempotents of A_P^G whose image in $A(P)$ is nonzero. Since Br_P^A sends A_P^G onto $(A(P))_1^{\overline{N}_G(P)}$ (see (1.1)(2)), and since $A^G = A_P^G$, we have $(A(P))^{\overline{N}_G(P)} = (A(P))_1^{\overline{N}_G(P)}$, and the first assertion then results from classical theorems about lifting idempotents: an indecomposable summand M with vertex P of $\mathrm{Ind}_P^G(1_{\mathcal{O}P})$ corresponds to a primitive idempotent i of A_P^G such that $\mathrm{Br}_P^A(i) \neq 0$, which corresponds to the primitive idempotent $\mathrm{Br}_P(i)$ of $(A(P))^{\overline{N}_G(P)}$, which in turn corresponds to the summand $\mathrm{Br}_P^A(i) \cdot \mathrm{Ind}_1^{\overline{N}_G(P)}(1_\mathbf{F}) = \mathrm{Br}_P^M(M) = M(P)$ of $\mathrm{Ind}_1^{\overline{N}_G(P)}(1_\mathbf{F})$.

The remark about $S_P(G,\mathcal{O})$ results from the fact that, by definition, we have $(S_P(G,\mathcal{O})(P))^{\overline{N}_G(P)} \neq 0$. We may also notice that

$$S_P(G,\mathcal{O})(P) = S_1(\overline{N}_G(P),\mathbf{F}).$$

PROOF OF (3.2)(4). If E is any indecomposable projective $\mathbf{F}\overline{N}_G(P)$-module, the $\mathcal{O}G$-module $M(P,E)$ is, by definition, the indecomposable p-permutation $\mathcal{O}G$-module determined by the condition $(M(P,E))(P) = E$. It is clear that if $M(P,E)$ is a summand of M, then E is a summand of $M(P)$. We prove the converse. Let M be a p-permutation $\mathcal{O}G$-module and suppose that E is a summand of $M(P)$; set $A = \mathrm{End}_\mathcal{O}(M)$. By (3.3) $A(P) \simeq \mathrm{End}_\mathbf{F}(M(P))$, so, by hypothesis, $(A(P))_1^{\overline{N}_G(P)}$ contains a primitive idempotent i such that $i \cdot M(P) \simeq E$. By the theorem about lifting idempotents and by (1.1)(2), we see that A_P^G contains a primitive idempotent j such that $\mathrm{Br}_P^A(j) = i$. Thus $j \cdot M$ is an indecomposable summand of M with vertex P, and we have $(j \cdot M)(P) = i \cdot M(P) \simeq E$, hence $j \cdot M \simeq M(P,E)$.

The next statement establishes the link between some of our constructions and more classical objects.

(3.4) *Let M be an indecomposable p-permutation $\mathbf{F}G$-module with vertex P. Then the Green correspondent of M is the $\mathbf{F}N_G(P)$-module $M(P)$.*

Indeed, we denote by N the Green correspondence of M and prove that $N \simeq M(P)$. Since M has a trivial source (see (0.4)), N also has a trivial source, hence is a summand of $\mathrm{Ind}_P^{N_G(P)}(1_{\mathbf{F}P})$; so P acts trivially on N, from which we deduce that $N(P) = N$. But by definition of the Green correspondence, $\mathrm{Res}_{N_G(P)}^G(M) \simeq N \oplus N'$, where N' is a sum of indecomposable $\mathbf{F}N_G(P)$-modules with vertex strictly contained in P; by (1.3) we get $N'(P) = 0$, and

$$M(P) = (\mathrm{Res}_{N_G(P)}^G(M))(P) \simeq N.$$

Let us recall two important consequences of (3.2)(3), which may as well be considered as consequences of properties of the Green correspondence applied to p-permutation modules.

(3.5) *The reduction modulo $\mathfrak{p}$ defines a bijection between the set of isomorphism classes of p-permutation $\mathcal{O}G$-modules and the set of isomorphism classes of p-permutation $\mathbf{F}G$-modules.*

Indeed, we know by (1.1)(3) that $M(P) = \overline{M}(P)$.

(3.6) *There is a bijection between the set of isomorphism classes of indecomposable p-permutation $\mathcal{O}G$-modules and the set of G-conjugacy classes of pairs (P, E), where P is a p-subgroup of G and E is an indecomposable projective $\mathbf{F}\overline{N}_G(P)$-module.*

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