ON SCOTT MODULES AND $p$-PERMUTATION MODULES:
AN APPROACH THROUGH THE BRAUER MORPHISM

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ABSTRACT. Following Lluis Puig we give a presentation of the theory of $p$-
permutation modules (also called “trivial source modules”) by a systematic
use of the generalized Brauer morphism.

The aim of this paper is to present a somewhat new and self-contained treatment
of $p$-permutation modules and Scott modules by a systematic use of the Brauer
morphism, as suggested by L. Puig (private communication).

By “self-contained” we mean that only knowledge of basic facts of representation
theory is needed: elementary theory of vertices and sources as presented in [5]
(see also [4]), as well as classical results about lifting idempotents. Burry’s result
about the module-theoretic interpretation of the coefficients of lower defect groups
is obtained as a by-product of that presentation.

Let $G$ be a finite group, and let $\mathcal{O}$ be a commutative ring, complete for a discrete
valuation, with maximal ideal $p$ and residual field $F = \mathcal{O}/p$. We assume $F$ has
characteristic $p > 0$. Note that $\mathcal{O}$ may be equal to $F$.

(0.1) DEFINITION. Let $M$ be an $\mathcal{O}$-free $\mathcal{O}G$-module. We say that $M$ is a $p$-
permutation module if, whenever $P$ is a $p$-subgroup of $G$, there is an $\mathcal{O}$-basis of $M$
stabilized by $P$.

(0.2) PROPOSITION. (1) If $M$ and $M'$ are two $p$-permutation $\mathcal{O}G$-modules, so
are the modules $M \oplus M'$ and $M \otimes M'$.

(2) Let $H$ be a subgroup of $G$. If $M$ (resp. $N$) is a $p$-permutation $\mathcal{O}G$-module
(resp. $\mathcal{O}H$-module), then $\text{Res}_H^G(M)$ (resp. $\text{Ind}_H^G(N)$) is a $p$-permutation $\mathcal{O}H$-module
(resp. $\mathcal{O}G$-module).

(3) Any summand of a $p$-permutation module is a $p$-permutation module.

The first two assertions are obvious. Let us prove the third. Let $M$ be a $p$-
permutation $\mathcal{O}G$-module, and let $M'$ be a summand of $M$. If $P$ is a $p$-subgroup of $G$, then $\text{Res}_P^G(M')$ is a summand of $\text{Res}_P^G(M)$. By definition $\text{Res}_P^G(M)$ is a direct
sum of modules isomorphic to $\text{Ind}_{PQ}^P(\mathcal{O}_Q)$ ($Q$ subgroups of $P$). Thus the assertion
will result from the following lemma.

(0.3) LEMMA. Let $P$ be a $p$-group, let $Q$ be a subgroup of $P$. Then the $\mathcal{O}G$-
module $\text{Ind}_{Q}^P(\mathcal{O}_Q)$ is indecomposable.

That result is well known and a consequence of Green’s theorem about induced
modules (see e.g. [4, Theorem 3.8]). We present here an elementary proof due to
M. Cabanes. It suffices to check that $\text{Ind}_{Q}^P(\mathcal{O}_Q)$ is indecomposable. Since a $p$-
group always has nontrivial fixed points on a nontrivial $F$-vector space, it suffices

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to prove that \((\text{Ind}_Q^P(1_{FQ}))^P\) has dimension 1 over \(F\); that last property results from Frobenius reciprocity, since
\[
(\text{Ind}_Q^P(1_{FQ}))^P = \text{Hom}_{FP}(1_{FP}, \text{Ind}_Q^P(1_{FQ})) = \text{Hom}_{FQ}(1_{FQ}, 1_{FQ}) = F.
\]

The following characterization shows that \(p\)-permutation modules are in fact familiar objects.

(0.4) Let \(M\) be an indecomposable \(\mathcal{O}G\)-module. The module \(M\) is a \(p\)-permutation \(\mathcal{O}G\)-module if and only if one of the following holds:

(i) there exists a subgroup \(H\) of \(G\) such that \(M\) is isomorphic to a summand of \(\text{Ind}_H^G(1_{\mathcal{O}H})\):
(ii) \(M\) has trivial source.

By (0.2)(3) any summand of \(\text{Ind}_H^G(1_{\mathcal{O}H})\) (hence, any module with trivial source) is a \(p\)-permutation module. Conversely, suppose that \(M\) is a \(p\)-permutation module, and let \(P\) be a \(p\)-subgroup such that \(M\) is \(P\)-projective; then \(M\) is a summand of \(\text{Ind}_P^G \text{Res}_P^G(M)\). But by definition \(\text{Res}_P^G(M)\) is a direct sum of modules \(\text{Ind}_Q^P(1_{\mathcal{O}Q})\) (\(Q\) subgroups of \(P\)); thus there exists a subgroup \(Q\) of \(P\) such that \(M\) is a summand of \(\text{Ind}_Q^P(1_{\mathcal{O}Q})\). Moreover, if \(P\) is a vertex of \(M\), necessarily \(Q = P\), proving that \(M\) has a trivial source.

1. On the Brauer morphism. We first need to recall some definitions (see [1, 2, 5]). Whenever \(M\) is an \(\mathcal{O}G\)-module and \(H\) and \(H'\) are two subgroups of \(G\) such that \(H \subset H'\), we denote by \(\text{Tr}_{H'}^H\) the linear map from \(M^H\) (the set of fixed points of \(H\) in \(M\)) into \(M^{H'}\) defined by (see [5]) \(\text{Tr}_{H'}^H(x) = \sum g(x)\), where \(g\) runs over a complete set of representatives in \(H'\) of \(H'/H\). We set \(\text{M}_{H'}^H = \text{Tr}_{H'}^H(M^H)\).

Whenever \(P\) is a \(p\)-subgroup of \(G\), we put \(\mathcal{N}_G(P) = N_G(P)/P\) and denote by \(M(P)\) the \(F\mathcal{N}_G(P)-\text{module}\) defined by (see [1 or 2])
\[
M(P) = M_P / \left( \sum_Q M_Q^P + pM_P \right)
\]

where \(Q\) runs over the set of proper subgroups of \(P\). We call “Brauer morphism” the natural surjection \(\text{Br}_P^G : M^P \to M(P)\). The Brauer morphism is a morphism of \(\mathcal{O}\mathcal{N}_G(P)\)-modules.

We shall use the following easy results (see e.g. [1 or 2]).

(1.1) (1) Suppose \(A\) is an \(\mathcal{O}G\)-algebra. Then \(\text{Ker}(\text{Br}_P^G)\) is a two-sided ideal of \(A^P\), \(A(P)\) is an \(F\mathcal{N}_G(P)-\text{algebra}\), and \(\text{Br}_P^G\) is a morphism of \(\mathcal{O}\mathcal{N}_G(P)\)-algebras.

(2) We have
\[
\text{Tr}_1^{\mathcal{N}_G(P)} \circ \text{Br}_P^M = \text{Br}_P^M \circ \text{Tr}_G^P,
\]

and, in particular,
\[
\text{Br}_P^M(M^G_P) = (M(P))^1_1^{\mathcal{N}_G(P)}.
\]

(3) Suppose \(\text{Res}_P^G(M)\) is a permutation \(\mathcal{O}P\)-module with a basis \(X\) stabilized by \(P\). Then \(M(P)\) has for \(F\)-basis the set \(\text{Br}_P^M(C_X(P))\) in bijection with the set \(C_X(P)\) of fixed points of \(X\) under \(P\). Moreover, the \(F\mathcal{N}_G(P)\)-modules \(M(P)\) and \(\bar{M}(P)\) (where \(\bar{M} = M/pM\) are canonically isomorphic.
We also need some other properties.

(1.2) Let $M_1, M_2$, and $M$ be $O\!G$-modules, and suppose that $f: M_1 \times M_2 \to M$ is a bilinear map stable under $G$-action. Then $f$ induces a bilinear map $f_P: M_1(P) \times M_2(P) \to M(P)$ stable under $N_G(P)$-action such that
\[ f_P(\text{Br}_{M_1}^P(x_1), \text{Br}_{M_2}^P(x_2)) = \text{Br}_M^P(f(x_1, x_2)) \]
for all $x_1$ in $M_1$ and $x_2$ in $M_2$.

The proof of (1.2) is straightforward and left to the reader.

(1.3) Let $H$ be a subgroup of $G$ and let $P$ be a $p$-group of $G$. Let $M$ be an $H$-projective $O\!G$-module. Then $M(P) = 0$ unless $P$ is $G$-conjugate to a subgroup of $H$. In particular, if $N$ is any $O\!H$-module, then $(\text{Ind}_H^G(N))(P) = 0$ unless $P$ is $G$-conjugate to a subgroup of $H$.

Indeed, let $A = \text{End}_O(M)$. By Higman’s criteria, $\text{id}_M \in A_H^G$, but (see [1 or 5]) $A_H^G \subset \sum_{g \in G} \text{A}_{P \cap gHg^{-1}}^P$, from which it follows that $M^P \subset \sum_{g \in G} \text{M}_{F \cap gHg^{-1}}^P$, which establishes the assertion.

(1.4) Let $H$ be a subgroup of $G$, $P$ a $p$-subgroup of $G$, and $T_g(P,H)$ the set of all $g$ in $G$ such that $P^g \subset H$. Then as $F\!N_G(P)$-modules, we have an isomorphism
\[ (\text{Ind}_H^G(1_{O\!H}))(P) \cong \sum_g \text{Ind}_{N_{gHg^{-1}}^G(P)}^G(1_{F\!N_{gHg^{-1}}^G(P)}), \]
where $g$ runs over a set of representatives in $T_g(P,H)$ of $N_G(P) \setminus T_g(P,H)/H$.

Indeed, by Mackey’s theorem, we have
\[ \text{Res}_{N_G(P)}^G \text{Ind}_H^G(1_{O\!H}) \cong \sum_g \text{Ind}_{N_{gH}^G(P)}^G(1_{O\!(N_G(P) \cap gH)}), \]
where $g$ runs over a complete set of representatives of $N_G(P) \setminus G/H$. Now (1.4) follows from the fact that (by (1.3))
\[ (\text{Ind}_{N_{gH}^G(P)}^G(1_{O\!(N_G(P) \cap gH)}))(P) = 0 \quad \text{if} \quad P \not\subseteq gH, \]
and if $P \subset gH$, then $P$ acts trivially on $\text{Ind}_{N_{gH}^G(P)}^G(1_{O\!N_{gH}(P)})$.

2. Scott modules and Scott coefficients associated with a $p$-subgroup.

Alperin and Scott proved the following result.

(2.1) Let $P$ be a $p$-subgroup of $G$. There exists an indecomposable $p$-permutation $F\!G$-module with vertex $P$ denoted by $S_P(G,F)$, uniquely determined up to isomorphism by one of the following properties:

(i) $1_{F\!G}$ is isomorphic to a submodule of $S_P(G,F)$;

(ii) $1_{F\!G}$ is isomorphic to a quotient of $S_P(G,F)$.

Moreover, the module $S_P(G,F)$ is isomorphic to its dual and is a summand of $\text{Ind}_H^G(1_{F\!H})$ if and only if $P$ is $G$-conjugate to a Sylow $p$-subgroup of $H$.

The module $S_P(G,F)$ is called the Scott module of $G$ associated to $P$.

We obtain a new proof of that result as a by-product of the definition and the study (due to Lluis Puig and suggested to him by some ideas of J. A. Green in [6]), every $p$ permutation $O\!G$-module $M$, of an integer, denoted by $s_P(M)$ and called the “Scott coefficient of $M$ associated to $P$”, which will be shown to have the following property: the integer $s_P(M)$ is the number of factors isomorphic to $S_P(G,F)$ in a decomposition of $M$ into a direct sum of indecomposable modules.
(2.2) DEFINITION (L. PUIG). Let $M$ be a $p$-permutation $\mathcal{O}G$-module, and let $P$ be a $p$-subgroup of $G$. The Scott coefficient of $M$ associated with $P$ is

$$s_P(M) = \dim_F(\Br_P^M(M_G^P)).$$

By (1.1)(2) we see that

(2.3) We have

$$s_P(M) = \dim_F((M(P))_{\overline{N}_G(P)}).$$

The main results about Scott coefficients and Scott modules are, as we shall see, easy consequences of the following lemma.

(2.4) LEMMA. Let $P$ be a $p$-subgroup of $G$.

1. If $M$ and $M'$ are two $p$-permutation $\mathcal{O}G$-modules, $s_P(M \oplus M') = s_P(M) + s_P(M')$.

2. Let $M$ be a $p$-permutation $\mathcal{O}G$-module and let $M^* = \Hom_{\mathcal{O}}(M, \mathcal{O})$ be its dual. Then $s_P(M) = s_P(M^*)$.

3. Let $H$ be a subgroup of $G$. Then $s_P(\Ind_H^G(1_{\mathcal{O}H})) = 0$ unless $P$ is $G$-conjugate to a Sylow $p$-subgroup of $H$, in which case $s_P(\Ind_H^G(1_{\mathcal{O}H})) = 1$.

The first assertion is obvious. We prove the second. Let $X$ be an $\mathcal{O}$-basis of $M$ stable under $P$. Then the dual basis $X^*$ is an $\mathcal{O}$-basis of $M^*$ stable by $P$, and the operations of $P$ on $X$ and on $X^*$ are isomorphic. By (1.2) the canonical duality $M \times M^* \to \mathcal{O}$ induces a $\overline{N}_G(P)$-duality $M(P) \times M^*(P) \to \mathbb{F}$; hence, it induces an $\mathbb{F}\overline{N}_G(P)$-homomorphism $M^*(P) \to M(P)^*$ which obviously sends the basis $\Br_P^M(C_{X^*}(P))$ onto the dual basis of $\Br_P^M(C_X(P))$; hence, it is an isomorphism. Thus, $s_P(M^*)$ is equal to the rank of the endomorphism $\Tr_{\overline{N}_G(P)}$ of $M(P)^*$, which is equal to the rank of its transpose, i.e. the endomorphism $\Tr_{\overline{N}_G(P)}$ of $M(P)$; hence, $s_P(M) = s_P(M^*)$.

Let us now prove the third assertion. According to property (1.3), we see that $(\Ind_H^G(1_{\mathcal{O}H}))(P) = 0$ if $P$ is not conjugate to a subgroup of $H$. By Sylow theorems in $H$, we know that if $P^g$ is a Sylow $p$-subgroup in $H$ for some $g$ in $G$, then $P^g$ is also a Sylow $p$-subgroup of $H$ only for those $g'$ lying in $N_G(P)gH$. So by (1.4), we see that it suffices to prove that $(\Ind_H^G(1_{\mathcal{O}H}))^G$ has dimension 0 or 1 according to the fact that $H$ is, or is not, a $p'$-group: this is easy to check.

Now we can prove, following Puig's method, the main result about Scott modules.

(2.5) THEOREM (SCOTT, ALPERIN). Let $P$ be a $p$-subgroup of $G$.

1. There exists a unique indecomposable $p$-permutation $\mathcal{O}G$-module $S_P(G, \mathcal{O})$ such that $s_P(S_P(G, \mathcal{O})) \neq 0$. We have $s_P(S_P(G, \mathcal{O})) = 1$, and $S_P(G, \mathcal{O})$ is isomorphic to its dual.

2. If $H$ is a subgroup of $G$, then $S_P(G, \mathcal{O})$ is isomorphic to a summand of $\Ind_H^G(1_{\mathcal{O}H})$ if and only if $P$ is $G$-conjugate to a Sylow $p$-subgroup of $H$. In this case $S_P(G, \mathcal{O})$ is the unique indecomposable summand $M$ of $\Ind_H^G(1_{\mathcal{O}H})$ such that one of the following holds:

(i) $\Hom_{\mathcal{O}G}(1_{\mathcal{O}H}, M) \neq 0$,

(ii) $\Hom_{\mathcal{O}G}(M, 1_{\mathcal{O}G}) \neq 0$,

and we have $\Hom_{\mathcal{O}G}(1_{\mathcal{O}G}, M) \simeq \Hom_{\mathcal{O}G}(M, 1_{\mathcal{O}G}) \simeq \mathcal{O}$. 

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Proof of (2.5)(1). Any $p$-permutation $\mathcal{O}G$-module $M$ is a summand of some $\text{Ind}_Q^G(1_{\mathcal{O}Q})$ for a $p$-subgroup $Q$ of $G$ (see (0.4)). If $s_P(M) \neq 0$, by (2.4)(1), (3), we see that $P$ is $G$-conjugate to $Q$, and since $s_P(\text{Ind}_Q^G(1_{\mathcal{O}P})) = 1$, we see that $M$ is the unique indecomposable summand of $\text{Ind}_Q^G(1_{\mathcal{O}P})$ such that $s_P(M) \neq 0$; we have $s_P(M) = 1$, and by the unicity of $M$ it follows from (2.4)(2) that $M \cong M^*$. 

Proof of (2.5)(2). By (1) we see that $S_P(G, \mathcal{O})$ is a summand of $\text{Ind}_H^G(1_{\mathcal{O}H})$ if and only if $s_P(\text{Ind}_H^G(1_{\mathcal{O}H})) \neq 0$, so by (2.4)(3) if $P$ is $G$-conjugate to a Sylow $p$-subgroup of $H$. Moreover, it follows from Frobenius reciprocity that

$$\text{Hom}_{\mathcal{O}G}(1_{\mathcal{O}G}, \text{Ind}_P^G(1_{\mathcal{O}P})) \simeq \mathcal{O} \simeq \text{Hom}_{\mathcal{O}G}(\text{Ind}_P^G(1_{\mathcal{O}P}), 1_{\mathcal{O}G}).$$

Thus, in order to complete the proof of (2.5) and since $S_P(G, \mathcal{O}) \simeq S_P(G, \mathcal{O})^*$, it suffices to prove that $\text{Hom}_{\mathcal{O}G}(1_{\mathcal{O}G}, S_P(G, \mathcal{O})) \neq 0$. By (1.1)(2) we know that $(S_P(G, \mathcal{O}))^G$ is mapped onto $((S_P(G, \mathcal{O}))(P))_{1}^{N_{\mathcal{O}}(P)}$ by the Brauer morphism, and that last module is not zero by definition of $S_P(G, \mathcal{O})$ since its dimension is precisely $s_P(S_P(G, \mathcal{O}))$; so, in particular, $S_P(G, \mathcal{O})^G \neq 0$.

Let us notice now that Burry’s nice result [3] is an immediate consequence of that presentation.

(2.6) Corollary (Burry). Let $e$ be an idempotent of $Z\mathcal{F}G$, and let $P$ be a $p$-subgroup of $G$. Then the Brauer coefficient of $P$ associated with $e$ is the multiplicity of $S_P(G, F)$ as a summand of $(\mathcal{F}G)e$, where $G$ acts by conjugation.

Indeed, the coefficient $m_e(P)$ of $P$ associated with $e$ is (see [1, §II.1])

$$m_e(P) = \dim_F(\text{Br}_P((\mathcal{F}G)^e)),$$

i.e., precisely by Definition (2.2)

$$m_e(P) = s_P((\mathcal{F}G)e).$$

(2.7) Remark. Let $M$ be any $p$-permutation $G$-module, and let $\bar{M} = M/pM$. By (1.1)(3) we see that $s_P(M) = s_P(\bar{M})$. From the characterization of $S_P(G, \mathcal{O})$ given in (2.5) it follows then that $\bar{S}_P(G, \mathcal{O}) = S_P(G, F)$.

3. $p$-permutation modules through the Brauer morphism. As we shall see, the Brauer morphism is particularly convenient for the local study of $p$-permutation $\mathcal{O}G$-modules.

(3.1) Let $M$ be a $p$-permutation $\mathcal{O}G$-module, and let $P$ be a $p$-subgroup of $G$. Then $M(P)$ is a $p$-permutation $\mathcal{F}N_{\mathcal{O}}(P)$-module.

Indeed, let $Q$ be a Sylow $p$-subgroup of $N_G(P)$, and let $X$ be a $Q$-stable $\mathcal{O}$-basis of $M$. Then (see (1.1)(3)) $\text{Br}_P^M(G_X(P))$ is a $(Q/P)$-stable $\mathcal{F}$-basis of $M(P)$.

The next statement is an omnibus theorem giving the main properties of the Brauer morphism applied to $p$-permutation modules.

(3.2) Theorem. (1) The vertices of an indecomposable $p$-permutation $\mathcal{O}G$-module $M$ are the maximal $p$-subgroup $P$ of $G$ such that $M(P) \neq 0$.

(2) An indecomposable $p$-permutation $\mathcal{O}G$-module $M$ has vertex $P$ if and only if $M(P)$ is nontrivial and a projective $\mathcal{F}N_{\mathcal{O}}(P)$-module.

(3) The correspondence $M \rightarrow M(P)$ induces a bijection between the isomorphism classes of indecomposable $p$-permutation $\mathcal{O}G$-modules with vertex $P$ and the
isomorphism classes of indecomposable projective $\mathbf{FN}_G(P)$-modules. In particular, $(\mathcal{O}_G(P))(P)$ is the projective cover of $1_{\mathbf{FN}_G(P)}$.

(4) Let $M$ be a $p$-permutation $\mathcal{O}_G$-module, let $E$ be an indecomposable projective $\mathbf{FN}_G(P)$-module, and let $M(P, E)$ be the corresponding $p$-permutation $\mathcal{O}_G$-module with vertex $P$. Then $M(P, E)$ is a summand of $M$ if and only if $E$ is a summand of $M(P)$.

Part of the proof of (3.2) may be simplified by using the Green correspondence (see (3.4)). But in order for the presentation to be “self-contained”, we give an independent proof.

By (1.3) we see that it suffices to prove that $M(P) \neq 0$ whenever $P$ is a vertex of $M$. Suppose $M$ is a summand of $\text{Ind}_P^G(1_{\mathcal{O}_P})$. Then by Mackey’s theorem and by (0.3), $\text{Res}_P^G(M)$ is isomorphic to a direct sum of modules of type $\text{Ind}_P^G(1_{\mathcal{O}(P\cap gP)})$, where $g$ runs over a certain subset of $G$. But $M$ is a summand of $\text{Ind}_P^G\text{Res}_P^G(M)$, and since $P$ is a vertex of $M$, we see that in the set of $g$’s there is at least one element in $N_G(P)$. In other words, $1_{\mathcal{O}_P}$ is a summand of $\text{Res}_P^G(M)$, proving that $M(P) \neq 0$.

To prove (2)-(4) of (3.2), we need

(3.3) LEMMA (PUIG). Let $P$ be a $p$-subgroup of $G$, let $M$ be a $p$-permutation $\mathcal{O}_G$-module, and let $A = \text{End}_G(M)$. Then the natural operation of $A(P)$ over $M(P)$ induces an isomorphism of $\mathbf{FN}_G(P)$-algebras between $A(P)$ and $\text{End}_F(M(P))$. Moreover, for $a \in A^P$ and $x \in M^P$, we have

$$\text{Br}_P^A(a)(\text{Br}_P^M(x)) = \text{Br}_P^M(a(x)).$$

The natural bilinear map $A \times M \to M$ induces a bilinear map $A(P) \times M(P) \to M(P)$ which is stable under $\mathbf{FN}_G(P)$ (see (1.2)), hence an $\mathbf{FN}_G(P)$-morphism $A(P) \to \text{End}_F(M(P))$ satisfying the last condition of the lemma. Let us prove that this morphism is an isomorphism. Let $X$ be an $\mathcal{O}$-basis of $M$ stable under $P$. Then the set $X(A) = \{a_{x,y}x, y \in X\}$, where $a_{x,y}(z) = \delta_{y,z}x$ for $x, y, z$ in $X$, is a $P$-stable basis of $A$. It is clear that $C_{X(A)}(P) = \{a_{x,y}x, y \in C_X(P)\}$. For $x \in C_X(P)$ or $a \in C_{X(A)}(P)$, let us set $\bar{x} = \text{Br}_P^A(x)$ and $\bar{a} = \text{Br}_P^M(a)$. Then we have $\bar{a}_{x,y}(\bar{z}) = \delta_{y,z}x$ for $x, y, z$ in $C_X(P)$, which proves (see (1.1)(3)) that the morphism $A(P) \to \text{End}_F(M(P))$ is an isomorphism.

PROOF OF (3.2)(2). Suppose first that $M$ is an indecomposable $p$-permutation $\mathcal{O}_G$-module such that $M(P)$ is a nontrivial $\mathbf{FN}_G(P)$-projective module; let us denote $\text{End}_\mathcal{O}(M)$ by $A$. Then by (3.3) and the Higman criteria,

$$A(P)^{\mathbf{FN}_G(P)} = (A(P))^{\mathbf{FN}_G(P)}_1.$$ But $(A(P))^{\mathbf{FN}_G(P)}_1$ is the image, through the Brauer morphism $\text{Br}_P^A$, of $A^G$ (see (1.1)(2)). Now by results about lifting idempotents, since $\text{id}_M$ is the unique nonzero idempotent of $A^G$, we deduce that $\text{id}_M \in A^G_1$, proving that $M$ is $P$-projective and $M$ has $P$ as a vertex by (3.2)(1).

Conversely, if $M$ has vertex $P$, we have $\text{id}_M \in A^G_1$, so $\text{id}_{M(P)} \in (A(P))^{\mathbf{FN}_G(P)}_1$, which (with (3.3)) proves that $M(P)$ is a projective $\mathbf{FN}_G(P)$-module.

PROOF OF (3.2)(3). As a consequence of (1.4) we have

$$(\text{Ind}_P^G(1_{\mathcal{O}_P}))(P) = \text{Ind}_1^{\mathbf{FN}_G(P)}(1_F).$$
The indecomposable $p$-permutation $\mathcal{O}G$-modules with vertex $P$ correspond to the summands of $\text{Ind}_G^F(1_G P)$ with vertex $P$; the indecomposable projective $F\mathcal{N}_G(P)$-modules correspond to summands of $\text{Ind}_1^{\mathcal{N}_G(P)}(1_F)$. Setting $A = \text{End}_G(\text{Ind}_G^F(1_G P))$ by (3.3) we know that $A(P) \simeq \text{End}_F(\text{Ind}_1^{\mathcal{N}_G(P)}(1_F))$ as $F\mathcal{N}_G(P)$-algebras. Now the summands of $\text{Ind}_G^F(1_G P)$ with vertex $P$ correspond to the primitive idempotents of $A_P^G$ whose image in $A(P)$ is nonzero. Since $\text{Br}_P^A$ sends $A_P^G$ onto $(A(P))^\mathcal{N}_G(P)$ (see (1.1)(2)), and since $A^G = A_P^G$, we have $(A(P))^\mathcal{N}_G(P) = (A(P))^1\mathcal{N}_G(P)$, and the first assertion then results from classical theorems about lifting idempotents: an indecomposable summand $M$ with vertex $P$ of $\text{Ind}_G^F(1_G P)$ corresponds to a primitive idempotent $i$ of $A_P^G$ such that $\text{Br}_P^A(i) \neq 0$, which corresponds to the primitive idempotent $\text{Br}_P^A(i)$ of $(A(P))^\mathcal{N}_G(P)$, which in turn corresponds to the summand $\text{Br}_P^A(i) \cdot \text{Ind}_1^{\mathcal{N}_G(P)}(1_F) = \text{Br}_P^M(M) = M(P)$ of $\text{Ind}_1^{\mathcal{N}_G(P)}(1_F)$.

The remark about $\text{Sp}(G, \mathcal{O})$ results from the fact that, by definition, we have $\text{Sp}(G, \mathcal{O})(P) = \text{Sp}(\mathcal{N}_G(P), F)$.

PROOF OF (3.2) (4). If $E$ is any indecomposable projective $F\mathcal{N}_G(P)$-module, the $\mathcal{O}G$-module $M(P, E)$ is, by definition, the indecomposable $p$-permutation $\mathcal{O}G$-module determined by the condition $(M(P, E))(P) = E$. It is clear that if $M(P, E)$ is a summand of $M$, then $E$ is a summand of $M(P)$. We prove the converse. Let $M$ be a $p$-permutation $\mathcal{O}G$-module and suppose that $E$ is a summand of $M(P)$; set $A = \text{End}_\mathcal{O}(M)$. By (3.3) $A(P) \simeq \text{End}_F(M(P))$, so, by hypothesis, $(A(P))^\mathcal{N}_G(P)$ contains a primitive idempotent $i$ such that $i \cdot M(P) \simeq E$. By the theorem about lifting idempotents and by (1.1)(2), we see that $A_P^G$ contains a primitive idempotent $j$ such that $\text{Br}_P^A(j) = i$. Thus $j \cdot M$ is an indecomposable summand of $M$ with vertex $P$, and we have $(j \cdot M)(P) = i \cdot M(P) \simeq E$, hence $j \cdot M \simeq M(P, E)$.

The next statement establishes the link between some of our constructions and more classical objects.

(3.4) Let $M$ be an indecomposable $p$-permutation $FG$-module with vertex $P$. Then the Green correspondent of $M$ is the $FN_G(P)$-module $M(P)$.

Indeed, we denote by $N$ the Green correspondence of $M$ and prove that $N \simeq M(P)$. Since $M$ has a trivial source (see (0.4)), $N$ also has a trivial source, hence is a summand of $\text{Ind}_P^{N_G(P)}(1_F)$; so $P$ acts trivially on $N$, from which we deduce that $N(P) = N$. But by definition of the Green correspondence, $\text{Res}_P^{N_G(P)}(M) \simeq N \oplus N'$, where $N'$ is a sum of indecomposable $F\mathcal{N}_G(P)$-modules with vertex strictly contained in $P$; by (1.3) we get $N'(P) = 0$, and $M(P) = (\text{Res}_P^{N_G(P)}(M))(P) \simeq N$.

Let us recall two important consequences of (3.2)(3), which may as well be considered as consequences of properties of the Green correspondence applied to $p$-permutation modules.

(3.5) The reduction modulo $p$ defines a bijection between the set of isomorphism classes of $p$-permutation $\mathcal{O}G$-modules and the set of isomorphism classes of $p$-permutation $FG$-modules.

Indeed, we know by (1.1)(3) that $M(P) = \overline{M}$. 

(3.6) There is a bijection between the set of isomorphism classes of indecomposable p-permutation $\mathcal{O}G$-modules and the set of $G$-conjugacy classes of pairs $(P, E)$, where $P$ is a $p$-subgroup of $G$ and $E$ is an indecomposable projective $\mathbb{F}N_G(P)$-module.

REFERENCES

5. J. A. Green, Some remarks on defect groups, Math. Z. 107 (1968), 133–150.