CHARACTERIZATIONS OF NOETHERIAN AND HEREDITARY RINGS

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Abstract. We characterize the left Noetherian rings by the existence of decompositions of left modules into direct sums of an injective submodule and a submodule containing no injective submodule except 0. We also prove that a left Noetherian ring is left hereditary iff the suspension of each left ideal (see [7]) is injective or, equivalently, the above decomposition is unique.

0. Let $A$ be an Abelian group. Then $A = D \oplus B$, where $D$ is divisible and $B$ is reduced in the sense of Kaplansky [8]; that is, $B$ has no divisible submodule except 0. If this holds for left modules over a unitary ring $R$, with "injective" and "$i$-reduced" replacing "divisible" and "reduced", respectively, then $R$ must be left Noetherian. The proof requires the theorem of Eilenberg-Bass (see, e.g., Theorem 20.1, p. 111, of [2]) which characterizes left Noetherian by the requirement that direct sums of injective left modules are injective. Furthermore, if the decomposition, which will be referred to as an $i$-decomposition, is unique in the sense that $D$ is the unique injective submodule of $A$ for which $A = D \oplus B$, with $B$ $i$-reduced, then every sum of injective submodules is injective, and hence $R$ must be left hereditary by a theorem of Matlis [9]. At the same time, it will be shown that a left Noetherian ring is left hereditary iff the suspension of each left ideal is injective.

1. Throughout this paper, $R$ denotes a unitary ring.

Theorem 1. The following conditions are equivalent:

(i) $R$ is left Noetherian.

(ii) The direct sum of any countably infinite family of injective left $R$-modules is injective.

(iii) Any left $R$-module admits an $i$-decomposition.

(iv) Every injective left $R$-module is a direct sum of indecomposable left $R$-modules.

Proof. The equivalence (i) $\iff$ (ii) is known, and due to Eilenberg-Bass. The implication (i) $\implies$ (iii) is immediate; and (i) $\iff$ (iv) is the theorem of Matlis [9] and Papp [10]. Let us prove (iii) $\implies$ (ii). Let $\{S_n; n = 1, 2, 3, \ldots\}$ be a countably infinite
family of injective left $R$-modules. Let $D$ and $B$ realize an $i$-decomposition of $S = \bigoplus_{n=1}^{\infty} S_n$. For each positive integer $k$, $T_k = \bigoplus_{n=1}^{k} S_n$ is an injective submodule of $S$. $B$ being $i$-reduced, we must have $E(T_k \cap D) = T_k$ by which we deduce that $T_k \subseteq D$, where $E(T_k \cap D)$ denotes the injective hull (included in $T_k$) of $T_k \cap D$. In this way, $S = D$, which is injective. Q.E.D.

2. For convenience, we will "borrow" some notation from the homotopy theory of modules introduced by B. Eckmann and P. Hilton (for more details, see [7, Chapter 13; 6 and 5]). For any left $R$-module $A$, we define the "suspension" of $A$ to be $S(A) = E(A)/A$, where $E(A)$ denotes the injective hull of $A$. Let $f: A \to B$ be a morphism of left $R$-modules; we denote by $f \simeq 0$ the fact that $f$ can be extended to any module containing $A$. We write $\tilde{\pi}(A, B) = 0$ if, for any $f: A \to B, f \simeq 0$. We say that a left $R$-module $B$ is FG-injective if $\tilde{\pi}(A, B) = 0$ for any finitely generated left $R$-module $A$. The following properties hold:

1. $f \simeq 0$ iff $f$ can be extended to the injective hull $E(A)$ of $A$ or, equivalently, to any injective extension $E$ of $A$.

2. If $f = g + h$, then $g \simeq 0$ and $h \simeq 0$ imply $f \simeq 0$.

3. If $f = \alpha \circ \beta$, then either $\alpha \simeq 0$ or $\beta \simeq 0$ implies $f \simeq 0$.

4. For any injective extension $E$ of $A$, $S(A)$ is injective (or FG-injective, or FP-injective) iff $E/A$ is.

5. If $\tilde{\pi}(I, B) = 0$ for each left ideal $I$ of $R$, then $B$ is injective.

6. $B$ is FP-injective iff $\tilde{\pi}(A, B) = 0$ for any finitely presented left $R$-module $A$.

Lemma. Assume that $R$ satisfies:

(*) For any left ideal $I$ of $R$, $S(I)$ is FG-injective.

Then the sum of any FG-injective submodules of a left $R$-module is FG-injective and hence FP-injective.

Proof. Let $\{V_i; i \in J\}$ be a family of FG-injective submodules of a left $R$-module; we need to show that $\sum_{i \in J} V_i$ is FG-injective. We may reduce the problem to the case when $J$ is finite, and so it suffices to consider $J = \{1, 2\}$. For any positive integer $m$, let $\hat{\mathcal{S}}^m$ be the family of all $R$-modules which can be spanned by $m$ elements. We will prove the following by induction on $m$:

$\tilde{\pi}(F, V_1 + V_2) = 0$ for any $F \in \hat{\mathcal{S}}^m$.

Firstly, for $m = 1$, let $F = Ra \in \hat{\mathcal{S}}^1$ and let $f: F \to V_1 + V_2. f(a) = v_1 + v_2$ for some $v_i \in V_i$, $i = 1, 2$. Since $V_i$ is FG-injective, there exists $\alpha_i: E(R) \to V_i$ such that $\alpha_i(r) = rv_i$ for all $r \in R$. Let $I = \{r \in R; r(v_1 + v_2) = 0\};$ and set $S = E(R)/I$ which is FG-injective by (*). Define $\beta: F \to S$ and $\alpha: S \to V_1 + V_2$ by $\beta(ra) = [r]$ in $S = E(R)/I$ and $\alpha([\tilde{r}]) = \alpha_1(\tilde{r}) + \alpha_2(\tilde{r})$ for $r \in R$ and $\tilde{r} \in E(R).$ We get $f = \alpha \circ \beta$. Now $\beta \simeq 0$, because $S$ is FG-injective. So $f \simeq 0$, and thus (*) holds for $m = 1$.

Next, we prove (*) for $m > 1$, assuming that (*) is true if $m$ is replaced by $m - 1$. Let $F = Ra_1 + Ra_2 + \cdots + Ra_m \in \hat{\mathcal{S}}^m$. We need show that $f \simeq 0$ for $f: F \to V_1 + V_2$. Consider $f_1 = f/Ra_1$. By the inductive hypothesis, $f_1 \simeq 0$, i.e. there is $f'_1: E(F) \to V_1 + V_2$ such that $f'_1/Ra_1 = f_1$. Let $g = f'_1/F$ and let $h = f - g$. Then
h(a_1) = 0, and so there exists \( \alpha: H = F/(Ra_1) \to V_1 + V_2 \) such that \( h = \alpha \circ p \), where \( p: F \to H = F/(Ra_1) \) is the canonical projection. Clearly, \( H \cong \mathbb{F}^m \); again by the inductive hypothesis, \( \alpha \approx 0 \), whence \( h \approx 0 \). But obviously, \( g \approx 0 \); thus \( f = g + h \approx 0 \). Q.E.D.

As a consequence, if \( R \) is a left Noetherian ring verifying (\(*\)), then \( \sum_{i \in J} V_i \) is injective when \( V_i \ (i \in J) \) all are.

We also observe that if \( \{ V_i; \ i \in J \} \) is a family of injective submodules of a left module over a left Noetherian ring, and if \( \sum_{i \in J} V_i \) admits a unique \( i \)-decomposition, then \( \sum_{i \in J} V_i \) is injective. At this stage, using a theorem of Matlis [9] (see, e.g., Proposition 10 of [3]), we may conclude

**Theorem 2.** The following conditions are equivalent:

(i) \( R \) is left Noetherian and left hereditary.

(ii) \( R \) is left Noetherian and satisfies (\(*\)).

(iii) Any left \( R \)-module admits a unique \( i \)-decomposition.

(iv) Every sum of injective submodules of a left \( R \)-module is injective.

The equivalence (i) \( \Leftrightarrow \) (ii) includes Corollary 14.B of [4].

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**REFERENCES**


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