ON THE EXPONENT OF NORM RESIDUE GROUPS

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Abstract. We compute the exponent of some norm residue groups in the number theoretic case (global fields). We use the method of Galois cohomology and the theory of the Brauer group over a global field.

Let $n$ be a natural number and let $K/k$ be a Galois extension of arbitrary fields with bicyclic group $G = \mathbb{Z}/n \times \mathbb{Z}/n$ generated by $s$ and $t$. By $H^r(G, K^*)$, $r \in \mathbb{Z}$, we mean the Tate cohomology groups of $G$ with coefficients in the multiplicative group of $K$. We consider the following diagram of subfields of $K$ and relative norms:

$$
\begin{array}{c}
K(s) = K^* \\
\downarrow N_1 \\
N \\
\downarrow N_2 \\
K_1 = K^{(t)}
\end{array}
\quad
\begin{array}{c}
N_2 \\
\downarrow N_1 \\
K
\end{array}
\quad
\begin{array}{c}
K_2 = K^{(t)}
\end{array}
$$

We use the known structure of the third Galois cohomological groups (see [4 and 5]):

(1) $H^3(G, K^*) = Z^3/B^3 \equiv N_1K_2^* \cap N_2K_1^*/NK^*$,

where the 3-cocycles are

$Z^3 = \{ c = (a, b) \in K_1^* \times K_2^* | N_2a = b \}$

and the 3-coboundaries are

$B^3 = \{ c = (a, b) \in K_1^* \times K_2^* | \text{there exists} \ (x, y, z) \in K_1^* \times K_2^* \times K^* \text{such that} \ t(x)N_1z = ax, s(y) = byN_2z \}.$

The sets $Z^3$ and $B^3$ are multiplicative groups through the operation $(a, b)(a', b') = (aa', bb')$. The isomorphism in (1) is induced by the map $Z^3 \to N_1K_2^* \cap N_2K_1^*/NK^*$ which sends $c = (a, b)$ to $N_1a = N_1b^{-1} \mod NK^*$.

The Diophantine equation $N(x) = a^n, a \in k^*$, $x$ an indeterminate with value in $K$, is of interest.

**Lemma.** Let $K/k$ be a Galois extension of arbitrary fields with group $G = \mathbb{Z}/n \times \mathbb{Z}/n$, $n \in \mathbb{N}$. If $H^3(G, K^*) = 0$, then $k^*/NK^*$ has exponent at most $n$, that is, every $n$th power of $k^*$ is a norm.

**Proof.** For all $a \in k^*$, the 3-cocycle $c = (a, a^{-1})$ is a coboundary. Hence the equation $N(x) = a^n$ has a solution.
It is possible to improve on this.

**Theorem 1.** Let $K/k$ be a Galois extension of global fields with group $G = \mathbb{Z}/p \times \mathbb{Z}/p$, $p$ an odd prime. Then the abelian group $k^*/Nk^*$ is of exponent $p$.

**Proof.** As $k^*/Nk^* = H^0(G, K^*)$ has exponent at most $p^2$, it suffices to show that $a^p = N(x)$ has a $k$-rational point for every $a \in k^*$. The equation $N(x) = a^p$ can be written as $N_1(aN_k(x^{-1})) = 1$. Using Hilbert 90, it suffices to find $d \in K^*$, $x \in K^*$ such that

$$a = N_2(x)s(d)d^{-1}.$$  

We interpret this equation in terms of central simple algebras. We introduce the cyclic crossed products $A = (K/K^2, a)$, $B = (K_1/k, a)$ and $C = (K/K^2, d)$. In the Brauer group, we have the following similarity (square brackets denote similarity):

$$[A] = [B \otimes_k K^2] \quad [3, (29.13)].$$

We write $C'$ for the algebra $C$ with the new $K_2$-module structure defined by $k \cdot d = s(k)d$, $k \in K^2$, $d \in C$. One can show that $C' = (K/K^2, s(d))$. We observe that solving (2) is the same as finding a central simple $K_2$-algebra $C$ such that

$$[A] = [C' \otimes_k C_{op}].$$

If $a \in N_2K^*$, there is nothing to show. Hence we suppose that $A$ is not the trivial algebra. We use the local invariants of Hasse for the description of the Brauer group of a global field [3, Chapter 8, or 2, Chapter VII]. We will need the exact sequence [3, (32.14)] for the extension $K/K^2$. As $A$ is obtained from $B$ by extension of the base field, we have from [2, Theorem 4, p. 113]

$$[A] = [B \otimes_k B].$$

Here $(A/w), (B/v)$ respectively $n_v$ denote the local invariants of the classes of $A, B$ respectively the local degree of the extension $(K_1)_w/k_v$. Thus the computation of the local invariants for the class of $A$ is reduced to a computation concerning the algebra $B$. Let $w$ be a place of $K^2$ and $v$ its restriction to $k$. Three cases are possible.

**Case 1.** If $w$ is an infinite place, it is clear that $(A/w) = 0$. Indeed, if $(A/w) = \frac{1}{2}$, then the local index 2 divides the exponent $p$ of the algebra $A$, which is impossible.

**Case 2.** If $w$ is a finite place invariant under the action of $s$, then $n_v = p$ and $(A/w) = (B/v)p$. But $(B/v) = 0$ or $s_v/p$, with $(s_v, p) = 1$, since $[B] \in Br(K_1/k)$ is of exponent 1 or $p$. It follows that $(A/w) = 0$.

**Case 3.** If the finite place $w$ is not invariant under $s$, we have $p$ places $w = w_1, w_2, \ldots, w_p$ above $v$ and $n_v = 1$. It follows that $(A/w_i) = (B/v) = 0$ or $s_v/p$ with $(s_v, p) = 1$.

As the next step, we construct a class $[C]$ by giving its local invariants. If $(A/w) = 0$, we put $(C/w) = 0$. We remark that $s$ is transitive on $w_1, w_2, \ldots, w_p$ and that $(C^s/w^s) = (C/w)$. If $(A/w_i) = s_v/p$ for $i = 1, \ldots, p$, we take the sequence of local invariants

$$\{(C/w_1), (C/w_2), \ldots, (C/w_p)\} = \{s_v/p, 2s_v/p, \ldots, (p-1)s_v/p, 0\}$$
such that after appropriate numbering of the \( w_i \)'s, we have
\[
\left\{(C^3/w_1), \ldots, (C^3/w_p)\right\} = \{2s_o/p, 3s_o/p, \ldots, 0, s_o/p\}.
\]
As \( \sum_{w_i}(C/w_i) = (p - 1)s_o/2 \equiv 0 \mod{\mathbb{Z}} \) if \( p \) is odd, the class \([C]\) is uniquely determined. The field \( K \) splits \( C \), since if \( (C/w) \neq 0 \), then \( (A/w) \neq 0 \), and so \( K \) has local degree \( p \) at \( w \). From the theory of crossed products and the fact that \( Br(K/K_2) \cong H^2(\langle i \rangle, K^*) \cong K^*_2/NK^* \), there exists \( d \in K^*_2 \) such that \( C = (K/K_2, d) \). By construction, the class of \( C \cong K^*_2 \) possesses the same local invariants as the class of \( A \). It follows that the algebra \( (K/K_2, a^{-1}s(d)d^{-1}) \), which is similar to the algebra \( A \cong K^*_2 \), is a matrix algebra. Therefore we have \( a^{-1}s(d)d^{-1} \in N_2K^* \) and the proof is complete.

**Remark.** The above proof is not valid if we omit the assumption \( p \) is odd. In fact there are biquadratic bicyclic extensions \( K/k \) such that \( k^*/NK^* \) is of exponent 4 as we will show below. This illustrates once more the difference in number theory between 2 and the other primes.

**Theorem 2.** Let \( K/k \) be a Galois extension of global fields with group \( G = \mathbb{Z}/2 \times \mathbb{Z}/2 \). Then we have:

(1) If \( H^3(G, K^*) = 0 \), then \( k^*/NK^* \) is of exponent 2.
(II) If \( H^3(G, K^*) \neq 0 \), then \( k^*/NK^* \) is of exponent 4.

**Proof.** In view of the lemma, it remains to show (II). The first author has computed that \( H^3(G, K^*) = k^*/\prod_{i=1}^3 N_i K_i^* \), where \( N_3 \) is the norm from \( K_3 = K^{(st)} \) to \( k \). (Details of proof will appear elsewhere; a connection with the Hasse problem is given below.) On the other side, we have \( \prod_{i=1}^3 N_i K_i^* = \{ x \in k^* | x^2 \in NK^* \} \) [1, Exercise 5, p. 360]. Since \( H^3(G, K^*) \neq 0 \), there exists \( x \in k^* \) with \( x^2 \notin NK^* \). We are done.

**Remark.** One can show that \( H^3(G, K^*) = N_1 K_2^* \cap N_2 K_1^*/NK^* \) is equal to the group \( \{\text{local norms}\}/\{\text{global norms}\} \). The question whether a local norm is equal to a global norm is known as the Hasse problem. In the biquadratic case, consider the explicit isomorphism \( N_1 K_2^* \cap N_2 K_1^*/NK^* \cong k^*/\prod_{i=1}^3 N_i K_i^* \), which sends a class \( N_1a = N_2b^{-1} \mod{NK^*} \) to the class \( a\prod_{i=1}^3 N_i K_i^* \mod{\prod_{i=1}^3 N_i K_i^*} \), where \( d \) satisfies \( ab = st(d)d^{-1} \). Using this isomorphism and Exercise 5 in [1], the first author has produced an algorithm which solves the Hasse problem in this particular case (details will appear elsewhere).

**References**


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