A NOTE ON THE ABSOLUTE CONVERGENCE OF
LACUNARY FOURIER SERIES

N. V. PATEL AND V. M. SHAH

Abstract. P. B. Kennedy [3] studied lacunary Fourier series whose generating functions are of bounded variation on a subinterval I of \([-\pi, \pi]\) and satisfy a Lipschitz condition of order \(\alpha\) on I. We show that the conclusion of one of his theorems on the absolute convergence of Fourier series remains valid when the function is merely of bounded \(r\)th variation in I and belongs to a class \(\text{Lip}(\alpha, p)\) in I. Our results also generalize three theorems of S. M. Mazhar [4].

1. Introduction. Let

\[
\sum_{k=1}^{\infty} \left( a_{n_k} \cos n_k x + b_{n_k} \sin n_k x \right)
\]

be the Fourier series of a \(2\pi\)-periodic function \(f \in L[-\pi, \pi]\), where \(\{n_k\} (k \in \mathbb{N})\) is a strictly increasing sequence of natural numbers such that

\[
n_{k+1} - n_k \to \infty \quad \text{as} \quad k \to \infty.
\]

P. B. Kennedy [3] proved the following theorem which recalls a well-known result of Zygmund [7, p. 136].

Theorem A [3, Theorem V(iv)]. If

(i) \(\{n_k\}\) satisfies (1.2),
(ii) \(f\) is of bounded variation in a subinterval \(I = [x_0 - \delta, x_0 + \delta]\) of \([-\pi, \pi]\), and
(iii) \(f \in \text{Lip} \alpha\) in I, where \(0 < \alpha < 1\), then

\[
\sum_{k=1}^{\infty} \left( |a_{n_k}| + |b_{n_k}| \right) < \infty.
\]

In [4, Theorem 1] S. M. Mazhar generalized a theorem of Noble [5], obtaining a variant of Theorem A in which (i) is replaced by a stronger condition, while (ii) is replaced by the weaker condition that \(f\) is of bounded \(r\)th variation. In this paper we prove a generalization of both theorems. The classes of functions we consider are defined below.

Received by the editors September 20, 1983 and, in revised form, March 20, 1984.
1980 Mathematics Subject Classification. Primary 42A28.
Key words and phrases. Lacunary Fourier series, absolute convergence, Lipschitz condition, bounded \(r\)th variation.

©1985 American Mathematical Society
0002-9939/85 $1.00 + $.25 per page

433
2. Definitions. (i) For $0 < \alpha \leq 1$ and $p \geq 1$, Lip($\alpha$, $p$, $I$) is defined to be the class of functions $f$ in $L[-\pi, \pi]$ such that

$$
\left( \int_I |f(x + h) - f(x)|^p \, dx \right)^{1/p} = O(|h|^\alpha) \quad \text{as } h \to 0.
$$

Evidently Lip $\alpha \subseteq$ Lip($\alpha$, $p$) for every $p \geq 1$.

(ii) Suppose a function $f$ is defined on an interval $[a, b]$ where $a \leq x_0 < x_1 < \cdots < x_{n-1} < x_n \leq b$. We write, as usual,

$$
\Delta^q f(x_i) = f(x_{i+1}) - f(x_i), \quad \Delta^q f(x_i) = \Delta^{q-1} f(x_{i+1}) - \Delta^{q-1} f(x_i) \quad (q \geq 2),
$$

so that

$$
\Delta^q f(x_i) = \sum_{r=0}^{q} (-1)^r \binom{q}{r} f(x_{i+q-r}).
$$

Then $f$ is said to be of bounded $r$th variation on $[a, b]$ (where $r$ is a positive integer) if there is a constant $M$ such that

$$
\sum_{i=0}^{n-r} |\Delta^r f(x_i)| \leq M
$$

whenever $x_0, x_1, \ldots, x_n$ are in arithmetical progression.

It is clear that every function of bounded variation is of bounded $r$th variation, but the converse is false, as is pointed out by Mazhar [4].

We prove the following theorems.

3. Results.

THEOREM 1. If

(i) $\{k_n\}$ satisfies (1.2),

(ii) $f$ is of bounded $r$th variation in $I$, and

(iii) $f \in$ Lip($\alpha$, $p$, $I$), where $0 < \alpha \leq 1, p > 2, \alpha p > 1$, then (1.3) holds.

This theorem clearly generalizes the results of Kennedy and Mazhar referred to in §1. It may also be compared to a theorem of Hardy and Littlewood [2] in which $r = 1, I = [-\pi, \pi]$, but lacunarity is not postulated.

The following extensions of Theorem 1 can also be proved; they contain Theorems 2 and 3 of [4].

THEOREM 2. If

(i) $\{k_n\}$ satisfies (1.2),

(ii) $f$ satisfies Theorem 1(ii), and

(iii) $f \in$ Lip($\alpha$, $p$, $I$), where $0 < \alpha \leq 1, p > 2$, then

$$
\sum_{k=1}^{\infty} \left( |a_{n_k}|^\beta + |b_{n_k}|^\beta \right) < \infty
$$

for every $\beta$ satisfying $2 > \beta > 2(p - 1)/(2p + \alpha p - 3)$. 
Theorem 3. Under the hypothesis of Theorem 2,
\[ \sum_{k=1}^{\infty} n_k^{p/2} |(a_{n_k} + b_{n_k})| < \infty \]
for \( \beta < (\alpha p - 1)/2(p - 1) \).

Observe that, for \( \beta = 1 \), Theorem 2 reduces to Theorem 1, and for \( \beta = 0 \), Theorem 3 also reduces to Theorem 1.

4. Proof of Theorem 1. We need the following lemmas. Lemma 1 is a special case of a very general theorem due to Paley and Wiener [6, Theorem XLII']. Lemma 2 is due to Kennedy [3, Lemma 1]. The result of Lemma 3 is given in the introduction of the paper [1] by Hardy and Littlewood.

Lemma 1. Suppose the function \( f \in L[-\pi, \pi] \) has Fourier series (1.1). If \( n_{k+1} - n_k \to \infty \) as \( k \to \infty \) and \( f \in L^2(I) \) for some subinterval \( I \) of \([-\pi, \pi] \), then \( f \in L^2[-\pi, \pi] \).

Lemma 2. (i) Let \( \{\lambda_k\} (\infty < k < \infty) \) be a sequence of real numbers satisfying \( \lambda_{k+1} - \lambda_k \to \infty (|k| \to \infty), \lambda_{k+1} - \lambda_k > 8\pi\delta^{-1} (\delta > 0) \).
(ii) Let \( \{A_k\} (\infty < k < \infty) \) be a sequence of complex numbers such that \( \sum_{-\infty}^{\infty} |A_k|^s |\lambda_k| < \infty (0 < s < 1) \).
(iii) Let
\[ \phi(s, x) = \sum_{-\infty}^{\infty} A_k s^{\lambda_k} e^{i\lambda_k x} \quad (0 < s < 1, -\infty < x < \infty), \]
(iv) Let
\[ \phi(x) = L^2 \lim_{s \to 1} \phi(s, x) \quad \text{in} \quad I = [x_0 - \delta, x_0 + \delta]. \]

Then
\[ \sum_{-\infty}^{\infty} |A_k|^2 \leq 8\delta^{-1} \int_I |\phi(x)|^2 dx. \]

Lemma 3. If \( f \in Lip(\alpha, p, I), p \geq 1 \), then \( f \in L^p(I) \).

Putting \( n_0 = 0, n_k = -n_{-k} (k < 0), C_{n_0} = 0, C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k}) (k > 0), \) and \( C_{n_k} = \overline{C}_{n_{-k}} (k < 0) \), we write (1.1) in the form
\[ \sum_{k=-\infty}^{\infty} C_k e^{in_k x}. \]

Since \( C_{n_k} \to 0 \) as \( k \to \infty \),
\[ \sum_{k=-\infty}^{\infty} |C_{n_k}|^s |n_k| < \infty \quad (0 < s < 1). \]

Put
\[ f(s, x) = \sum_{k=-\infty}^{\infty} C_k s^{\lambda_k} e^{in_k x} |n_k| \quad (0 < s < 1, -\infty < x < \infty). \]
Now, by Theorem 1(iii), \( f \in \text{Lip}(\alpha, 2, I) \) and, therefore, by Lemma 3, \( f \in L^2(I) \). Hence, by Lemma 1, \( f \in L^2[-\pi, \pi] \), and so, by a known theorem [7, p. 87], it follows that

\[
(4.2) \quad f(x) = L^2\lim_{s \to 1} f(s, x) \quad (|x| \leq \pi).
\]

Take integers \( K, j \) such that

\[
(4.3) \quad n_K > 2r\pi/\delta \quad \text{and} \quad 0 < j \leq \delta n_K/8\pi.
\]

In the following argument we let \( r \) be an even integer; the treatment for odd \( r \) is similar.

Put

\[
(4.4) \quad g_j(x) = \sum_{\nu = 0}^r (-1)^\nu \binom{r}{\nu} f\left(x + \frac{2j\pi}{n_K} + \frac{(r - 2\nu)\pi}{2n_K}\right)
\]

and

\[
Akj = C_{n_k}\left(2i \sin \frac{n_k\pi}{2n_K}\right)^r \exp\left(\frac{in_k2j\pi}{n_K}\right).
\]

Since \( |A_{kj}| \leq |C_{n_k}|2^r \), it follows from (4.1) that

\[
(4.5) \quad \sum_{k = -\infty}^{\infty} |A_{kj}|s^{\ln k} \leq \infty \quad (0 < s < 1).
\]

If

\[
(4.6) \quad g_j(s, x) = \sum_{k = -\infty}^{\infty} A_{kj}s^{\ln k} e^{in_kx} \quad (0 < s < 1, -\infty < x < \infty),
\]

we have

\[
g_j(s, x) = \sum_{\nu = 0}^r (-1)^\nu \binom{r}{\nu} f\left(s, x + \frac{2j\pi}{n_K} + \frac{(r - 2\nu)\pi}{2n_K}\right);
\]

from which, together with (4.2) and (4.4), we obtain

\[
(4.7) \quad g_j(x) = L^2\lim_{s \to 1} g_j(s, x) \quad \text{in } [x_0 - \frac{1}{2}\delta, x_0 + \frac{1}{2}\delta].
\]

We may assume that

\[
(4.8) \quad n_{k+1} - n_k > 16\pi\delta^{-1} \quad \text{for all } k.
\]

In view of (1.2), this can be achieved, if necessary, by adding to \( f(x) \) a polynomial in \( \exp(in_k x) \), a process which affects neither the hypothesis nor the conclusion of the theorem. Hence, putting \( I_1 = [x_0 - \frac{1}{2}\delta, x_0 + \frac{1}{2}\delta] \), we obtain from (4.5)-(4.8) and Lemma 2 that

\[
(4.9) \quad \sum_{k = -\infty}^{\infty} |A_{kj}|^2 \leq 16\delta^{-1} \int_{I_1} |g_j(x)|^2 \, dx.
\]

But

\[
|A_{kj}| = |C_{n_k}|2^r\left(\sin \frac{n_k\pi}{2n_K}\right)^r.
\]
Therefore (4.9) gives
\[
\sum_{n_k/2}^{n_k} 2^{2r} \left( \sin \frac{n_k \pi}{2n_k} \right)^{2r} |C_{n_k}|^2 \leq 16 \delta^{-1} \int_{I_1} |g_j(x)|^2 \, dx.
\]
But if \( \frac{1}{2} n_k \leq |n_k| \leq n_k \), then
\[
\left( \sin \frac{n_k \pi}{2n_k} \right)^{2r} \geq \left( \frac{1}{2} \right)^r.
\]
Therefore (4.10) reduces to
\[
\sum_{n_k/2}^{n_k} |C_{n_k}|^2 \leq \frac{16 \delta^{-1}}{2^r} \int_{I_1} |g_j(x)|^2 \, dx.
\]
Let \( p' = p - 1 \). Since \( p > 2 \), \( p' > 1 \). Define \( q \) by the equation \( 1/p' + 1/q = 1 \).
We now denote, by \( \Sigma_j \), summation over the range \( 0 < j < \lceil \delta n_k/8 \pi \rceil \) (where \( \lceil a \rceil \) represents the integral part of \( a \)). It then follows from (4.11) that
\[
\left\{ \sum_{n_k/2}^{n_k} |C_{n_k}|^2 \right\}^q \leq \frac{16 \delta^{-1} 2^{-r}}{\lceil \delta n_k/8 \pi \rceil} \sum_j \left( \int_{I_1} |g_j(x)|^2 \, dx \right)^q
\]
where the constant \( C \) depends on \( \delta \) and \( r \). \( C \) will henceforth denote positive constants not necessarily the same at each occurrence.

Since \( 2 = (p' + 1)/p' + 1/q \), an application of Hölder's inequality gives
\[
\left( \int_{I_1} |g_j(x)|^2 \, dx \right)^q = \left( \int_{I_1} |g_j(x)|^{(p' + 1)/p' + 1/q} \, dx \right)^q
\]
\[
\leq \left[ \left( \int_{I_1} |g_j(x)|^{(p' + 1)/p'} \, dx \right)^{1/p'} \left( \int_{I_1} |g_j(x)|^{1/q} \, dx \right)^{1/q} \right]^q
\]
\[
= \left( \int_{I_1} |g_j(x)|^p \, dx \right)^{q/p} \left( \int_{I_1} |g_j(x)|^q \, dx \right)^{q/q}.
\]
This, combined with (4.12), gives
\[
\left\{ \sum_{n_k/2}^{n_k} |C_{n_k}|^2 \right\}^q \leq \frac{C}{n_k} \sum_j \left( \int_{I_1} |g_j(x)|^p \, dx \right)^{q/p} \left( \int_{I_1} |g_j(x)|^q \, dx \right).
\]
Putting \( y = x + 2 j \pi / n_k + r \pi / 2 n_k \) in (4.4), we evidently have
\[
g_j(x) = \sum_{y = 0}^{r - 1} a_y \left( f \left( y - \frac{v \pi}{n_k} \right) - f \left( y - \frac{(v + 1) \pi}{n_k} \right) \right),
\]
where \( a_0, a_1, \ldots, a_{r - 1} \) are constants. Hence, by Minkowski's inequality and condition Theorem 1(iii),
\[
\left[ \int_{I_1} |g_j(x)|^p \, dx \right]^{1/p} \leq \sum_{y = 0}^{r - 1} |a_y| \left[ \int_{I_1} \left| f \left( y - \frac{v \pi}{n_k} \right) - f \left( y - \frac{(v + 1) \pi}{n_k} \right) \right|^p \, dx \right]^{1/p}
\]
\[
= O(n_k^{-a}).
\]
Also, for $|x - x_0| \leq \delta/2$ and integers $j$ and $K$ satisfying (4.3), the points $y = \nu \pi/n_K$ ($\nu = 0, 1, \ldots, r$) all lie in $[x_0 - \delta, x_0 + \delta]$. Therefore, if $V$ is the total $r$th variation of $f$ in $I$, it follows from Theorem 1(ii) that

$$\sum_j |g_j(x)| \leq V \quad \text{for } |x - x_0| \leq \frac{1}{2} \delta.$$ 

Hence, by (4.13),

$$\left( \sum_{n_K/2}^{n_K} |C_{n_k}|^2 \right)^{q} \leq C n_K^{-1} n_K^{-1 - apq/p'} V \delta = C n_K^{-1 - apq/p'},$$

which in turn gives

$$(4.14) \quad \sum_{n_K/2}^{n_K} |C_{n_k}|^2 \leq C n_K^{-1 - apq/p'}.$$ 

Let $m$ be a positive integer. Either the set of integers $k$ for which $2^m < |n_k| \leq 2^{m+1}$ is empty, or there is a member of this set, say $K = K(m)$, which has largest modulus; and in the latter case, the set is included in the set of $k$ for which $\frac{1}{2} n_K \leq |n_k| \leq n_K$. Moreover, if $m$ is large enough, $K$ satisfies (4.3). Hence, in either case it follows from (4.14) that

$$\sum_{2^m < |n_k| \leq 2^{m+1}} \left\{ |C_{n_k}|^2 \right\} \leq \sum_{n_K/2}^{n_K} \left\{ |C_{n_k}|^2 \right\} = O\left( n_K^{-1 - apq/p'} \right)$$

$$= O\left( 2^{m(1 - apq/p')} \right).$$

Also, by (4.8), the number of terms in the above summation is $O(2^m)$. Hence, by Cauchy's inequality,

$$\sum_{2^m < |n_k| \leq 2^{m+1}} |C_{n_k}| \leq \left( \sum_{2^m < |n_k| \leq 2^{m+1}} |C_{n_k}|^2 \right)^{1/2} \left( \sum_{2^m < |n_k| \leq 2^{m+1}} 1 \right)^{1/2}$$

$$= O\left( 2^{m/2(1 - apq/p')} \right),$$

so

$$(4.15) \quad \sum_{k = -\infty}^{\infty} |C_{n_k}| \leq C \sum_{m = 1}^{\infty} 2^{m(1 - apq)/p'}.$$ 

Since $ap > 1$, the series on the right side is convergent. This completes the proof of Theorem 1.

Theorems 2 and 3 can be proved in a similar manner.

Thanks are due to the referee for his useful comments.

REFERENCES


DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, THE MAHARAJA SAYAJIRAO UNIVERSITY OF BARODA, BARODA - 390 002, INDIA