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A NOTE ON THE ABSOLUTE CONVERGENCE OF LACUNARY FOURIER SERIES

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ABSTRACT. P. B. Kennedy [3] studied lacunary Fourier series whose generating functions are of bounded variation on a subinterval I of $[-\pi, \pi]$ and satisfy a Lispschitz condition of order α on I. We show that the conclusion of one of his theorems on the absolute convergence of Fourier series remains valid when the function is merely of bounded rth variation in I and belongs to a class $\mathrm{Lip}(\alpha, p)$ in I. Our results also generalize three theorems of S. M. Mazhar [4].

1. Introduction. Let

$$\sum_{k=1}^{\infty} \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right) \tag{1.1}$$

be the Fourier series of a 2π-periodic function $f \in L[-\pi, \pi]$, where $\{n_k\}$ $(k \in N)$ is a strictly increasing sequence of natural numbers such that

$$n_{k+1} - n_k \to \infty \quad \text{as } k \to \infty. \tag{1.2}$$

P. B. Kennedy [3] proved the following theorem which recalls a well-known result of Zygmund [7, p. 136].

THEOREM A [3, THEOREM V(iv)]. *If*
(i) $\{n_k\}$ *satisfies* (1.2),
(ii) f *is of bounded variation in a subinterval* $I = [x_0 - \delta, x_0 + \delta]$ *of* $[-\pi, \pi]$, *and*
(iii) $f \in \mathrm{Lip}\,\alpha$ *in* I, *where* $0 < \alpha < 1$, *then*

$$\sum_{k=1}^{\infty} \left(|a_{n_k}| + |b_{n_k}|\right) < \infty. \tag{1.3}$$

In [4, Theorem 1] S. M. Mazhar generalized a theorem of Noble [5], obtaining a variant of Theorem A in which (i) is replaced by a stronger condition, while (ii) is replaced by the weaker condition that f is of bounded rth variation. In this paper we prove a generalization of both theorems. The classes of functions we consider are defined below.

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2. Definitions. (i) For $0 < \alpha \leqslant 1$ and $p \geqslant 1$, $\mathrm{Lip}(\alpha, p, I)$ is defined to be the class of functions f in $L[-\pi, \pi]$ such that

$$\left(\int_I |f(x+h) - f(x)|^p \, dx\right)^{1/p} = O\left(|h|^\alpha\right) \quad \text{as } h \to 0.$$

Evidently $\mathrm{Lip}\,\alpha \subseteq \mathrm{Lip}(\alpha, p)$ for every $p \geqslant 1$.

(ii) Suppose a function f is defined on an interval $[a, b]$ where $a \leqslant x_0 < x_1 < \cdots < x_{n-1} < x_n \leqslant b$. We write, as usual,

$$\Delta^1 f(x_i) = f(x_{i+1}) - f(x_i), \qquad \Delta^q f(x_i) = \Delta^{q-1} f(x_{i+1}) - \Delta^{q-1} f(x_i) \quad (q \geqslant 2),$$

so that

$$\Delta^q f(x_i) = \sum_{\nu=0}^{q} (-1)^\nu \binom{q}{\nu} f(x_{i+q-\nu}).$$

Then f is said to be of bounded rth variation on $[a, b]$ (where r is a positive integer) if there is a constant M such that

$$\sum_{i=0}^{n-r} |\Delta^r f(x_i)| \leqslant M$$

whenever $x_0, x_1, \ldots, x_n$ are in arithmetical progression.

It is clear that every function of bounded variation is of bounded rth variation, but the converse is false, as is pointed out by Mazhar [**4**].

We prove the following theorems.

3. Results.

THEOREM 1. *If*

(i) $\{n_k\}$ *satisfies* (1.2),

(ii) f *is of bounded rth variation in* I, *and*

(iii) $f \in \mathrm{Lip}(\alpha, p, I)$, *where* $0 < \alpha \leqslant 1, p > 2, \alpha p > 1$, *then* (1.3) *holds.*

This theorem clearly generalizes the results of Kennedy and Mazhar referred to in §1. It may also be compared to a theorem of Hardy and Littlewood [**2**] in which $r = 1$, $I = [-\pi, \pi]$, but lacunarity is not postulated.

The following extensions of Theorem 1 can also be proved; they contain Theorems 2 and 3 of [**4**].

THEOREM 2. *If*

(i) $\{n_k\}$ *satisfies* (1.2),

(ii) f *satisfies Theorem* 1(ii), *and*

(iii) $f \in \mathrm{Lip}(\alpha, p, I)$, *where* $0 < \alpha \leqslant 1, p > 2$, *then*

$$\sum_{k=1}^{\infty} \left(|a_{n_k}|^\beta + |b_{n_k}|^\beta\right) < \infty$$

for every β *satisfying* $2 > \beta > 2(p-1)/(2p + \alpha p - 3)$.

THEOREM 3. *Under the hypothesis of Theorem* 2,

$$\sum_{k=1}^{\infty} n_k^{\beta/2}\left(|a_{n_k}| + |b_{n_k}|\right) < \infty$$

for $\beta < (\alpha p - 1)/2(p - 1)$.

Observe that, for $\beta = 1$, Theorem 2 reduces to Theorem 1, and for $\beta = 0$, Theorem 3 also reduces to Theorem 1.

4. Proof of Theorem 1. We need the following lemmas. Lemma 1 is a special case of a very general theorem due to Paley and Wiener [**6**, Theorem XLII′]. Lemma 2 is due to Kennedy [**3**, Lemma 1]. The result of Lemma 3 is given in the introduction of the paper [**1**] by Hardy and Littlewood.

LEMMA 1. *Suppose the function* $f \in L[-\pi, \pi]$ *has Fourier series* (1.1). *If* $n_{k+1} - n_k \to \infty$ *as* $k \to \infty$ *and* $f \in L^2(I)$ *for some subinterval* I *of* $[-\pi, \pi]$, *then* $f \in L^2[-\pi, \pi]$.

LEMMA 2. (i) *Let* $\{\lambda_k\}$ $(-\infty < k < \infty)$ *be a sequence of real numbers satisfying* $\lambda_{k+1} - \lambda_k \to \infty$ $(|k| \to \infty)$, $\lambda_{k+1} - \lambda_k > 8\pi\delta^{-1}$ $(\delta > 0)$.

(ii) *Let* $\{A_k\}$ $(-\infty < k < \infty)$ *be a sequence of complex numbers such that* $\sum_{-\infty}^{\infty}|A_k|s^{|\lambda_k|} < \infty$ $(0 < s < 1)$.

(iii) *Let*

$$\phi(s, x) = \sum_{-\infty}^{\infty} A_k s^{|\lambda_k|} e^{i\lambda_k x} \qquad (0 < s < 1, -\infty < x < \infty),$$

(iv) *Let*

$$\phi(x) = L^2\text{-}\lim_{s \to 1} \phi(s, x) \quad \text{in } I = [x_0 - \delta, x_0 + \delta].$$

Then

$$\sum_{-\infty}^{\infty} |A_k|^2 \leqslant 8\delta^{-1} \int_I |\phi(x)|^2 \, dx.$$

LEMMA 3. *If* $f \in \mathrm{Lip}(\alpha, p, I)$, $p \geqslant 1$, *then* $f \in L^p(I)$.

Putting $n_0 = 0$, $n_k = -n_{-k}$ $(k < 0)$, $C_{n_0} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$, and $C_{n_k} = \overline{C}_{n_{-k}}$ $(k < 0)$, we write (1.1) in the form

$$\sum_{k=-\infty}^{\infty} C_{n_k} e^{in_k x}.$$

Since $C_{n_k} \to 0$ as $k \to \infty$,

$$\sum_{k=-\infty}^{\infty} |C_{n_k}| s^{|n_k|} < \infty \qquad (0 < s < 1). \tag{4.1}$$

Put

$$f(s, x) = \sum_{k=-\infty}^{\infty} C_{n_k} e^{in_k x} s^{|n_k|} \qquad (0 < s < 1, -\infty < x < \infty).$$

Now, by Theorem 1(iii), $f \in \mathrm{Lip}(\alpha, 2, I)$ and, therefore, by Lemma 3, $f \in L^2(I)$. Hence, by Lemma 1, $f \in L^2[-\pi, \pi]$, and so, by a known theorem [7, p. 87], it follows that

$$f(x) = L^2\text{-}\lim_{s \to 1} f(s, x) \qquad (|x| \leqslant \pi). \tag{4.2}$$

Take integers K, j such that

$$n_K > 2r\pi/\delta \quad \text{and} \quad 0 \leqslant j \leqslant \delta n_K/8\pi. \tag{4.3}$$

In the following argument we let r be an even integer; the treatment for odd r is similar.

Put

$$g_j(x) = \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} f\left(x + \frac{2j\pi}{n_K} + \frac{(r-2\nu)\pi}{2n_K}\right) \tag{4.4}$$

and

$$A_{kj} = C_{n_k}\left(2i \sin \frac{n_k \pi}{2n_K}\right)^r \exp\left(\frac{in_k 2j\pi}{n_K}\right).$$

Since $|A_{kj}| \leqslant |C_{n_k}| 2^r$, it follows from (4.1) that

$$\sum_{k=-\infty}^{\infty} |A_{kj}| s^{|n_k|} < \infty \qquad (0 < s < 1). \tag{4.5}$$

If

$$g_j(s, x) = \sum_{k=-\infty}^{\infty} A_{kj} s^{|n_k|} e^{in_k x} \qquad (0 < s < 1, -\infty < x < \infty), \tag{4.6}$$

we have

$$g_j(s, x) = \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} f\left(s, x + \frac{2j\pi}{n_K} + \frac{(r-2\nu)\pi}{2n_K}\right);$$

from which, together with (4.2) and (4.4), we obtain

$$g_j(x) = L^2\text{-}\lim_{s \to 1} g_j(s, x) \quad \text{in } \left[x_0 - \tfrac{1}{2}\delta, x_0 + \tfrac{1}{2}\delta\right]. \tag{4.7}$$

We may assume that

$$n_{k+1} - n_k > 16\pi\delta^{-1} \quad \text{for all } k. \tag{4.8}$$

In view of (1.2), this can be achieved, if necessary, by adding to $f(x)$ a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem. Hence, putting $I_1 = [x_0 - \frac{1}{2}\delta, x_0 + \frac{1}{2}\delta]$, we obtain from (4.5)–(4.8) and Lemma 2 that

$$\sum_{k=-\infty}^{\infty} |A_{kj}|^2 \leqslant 16\delta^{-1} \int_{I_1} |g_j(x)|^2 \, dx. \tag{4.9}$$

But

$$|A_{kj}| = |C_{n_k}| 2^r \left(\sin \frac{n_k \pi}{2n_K}\right)^r.$$

Therefore (4.9) gives

$$\sum_{n_K/2}^{n_K} 2^{2r}\left(\sin\frac{n_k\pi}{2n_K}\right)^{2r}|C_{n_k}|^2 \leqslant 16\delta^{-1}\int_{I_1}|g_j(x)|^2\,dx. \tag{4.10}$$

But if $\frac{1}{2}n_K \leqslant |n_k| \leqslant n_K$, then

$$\left(\sin\frac{\pi n_k}{2n_K}\right)^{2r} \geqslant \left(\frac{1}{2}\right)^r.$$

Therefore (4.10) reduces to

$$\sum_{n_K/2}^{n_K}|C_{n_k}|^2 \leqslant \frac{16\delta^{-1}}{2^r}\int_{I_1}|g_j(x)|^2\,dx. \tag{4.11}$$

Let $p' = p - 1$. Since $p > 2$, $p' > 1$. Define q by the equation $1/p' + 1/q = 1$.

We now denote, by $\sum_j$, summation over the range $0 \leqslant j \leqslant [\delta n_K/8\pi]$ (where $[a]$ represents the integral part of a). It then follows from (4.11) that

$$\left\{\sum_{n_K/2}^{n_K}|C_{n_k}|^2\right\}^q \leqslant \frac{(16\delta^{-1}2^{-r})^q}{[\delta n_K/8\pi]}\sum_j\left\{\int_{I_1}|g_j(x)|^2\,dx\right\}^q \tag{4.12}$$

$$\leqslant \frac{C}{n_K}\sum_j\left\{\int_{I_1}|g_j(x)|^2\,dx\right\}^q,$$

where the constant C depends on δ and r. C will henceforth denote positive constants not necessarily the same at each occurrence.

Since $2 = (p'+1)/p' + 1/q$, an application of Hölder's inequality gives

$$\left\{\int_{I_1}|g_j(x)|^2\,dx\right\}^q = \left\{\int_{I_1}|g_j(x)|^{(p'+1)/p'+1/q}\,dx\right\}^q$$

$$\leqslant \left[\left(\int_{I_1}|g_j(x)|^{(p'+1)p'/p'}\,dx\right)^{1/p'}\left(\int_{I_1}|g_j(x)|^{1/q\cdot q}\,dx\right)^{1/q}\right]^q$$

$$= \left\{\int_{I_1}|g_j(x)|^p\,dx\right\}^{q/p'}\left\{\int_{I_1}|g_j(x)|dx\right\}.$$

This, combined with (4.12), gives

$$\left\{\sum_{n_K/2}^{n_K}|C_{n_k}|^2\right\}^q \leqslant \frac{C}{n_K}\sum_j\left\{\left(\int_{I_1}|g_j(x)|^p\,dx\right)^{q/p'}\int_{I_1}|g_j(x)|dx\right\}. \tag{4.13}$$

Putting $y = x + 2j\pi/n_K + r\pi/2n_K$ in (4.4), we evidently have

$$g_j(x) = \sum_{\nu=0}^{r-1}a_\nu\left\{f\left(y - \frac{\nu\pi}{n_K}\right) - f\left(y - \frac{(\nu+1)\pi}{n_K}\right)\right\},$$

where $a_0, a_1, \ldots, a_{r-1}$ are constants. Hence, by Minkowski's inequality and condition Theorem 1(iii),

$$\left[\int_{I_1}|g_j(x)|^p\,dx\right]^{1/p} \leqslant \sum_{\nu=0}^{r-1}|a_\nu|\left\{\int_I\left|f\left(y - \frac{\nu\pi}{n_K}\right) - f\left(y - \frac{(\nu+1)\pi}{n_K}\right)\right|^p dx\right\}^{1/p}$$

$$= O(n_K^{-\alpha}).$$

Also, for $|x - x_0| \leqslant \delta/2$ and integers j and K satisfying (4.3), the points $y - \nu\pi/n_K$ ($\nu = 0, 1, \ldots, r$) all lie in $[x_0 - \delta, x_0 + \delta]$. Therefore, if V is the total rth variation of f in I, it follows from Theorem 1(ii) that

$$\sum_j |g_j(x)| \leqslant V \quad \text{for } |x - x_0| \leqslant \frac{1}{2}\delta.$$

Hence, by (4.13),

$$\left\{ \sum_{n_K/2}^{n_K} |C_{n_k}|^2 \right\}^q \leqslant C n_K^{-1} n_K^{-\alpha p q/p'} V\delta = C n_K^{-1-\alpha p q/p'},$$

which in turn gives

$$\sum_{n_K/2}^{n_K} |C_{n_k}|^2 \leqslant C n_K^{-1/q-\alpha p/p'}. \tag{4.14}$$

Let m be a positive integer. Either the set of integers k for which $2^m < |n_k| \leqslant 2^{m+1}$ is empty, or there is a member of this set, say $K = K(m)$, which has largest modulus; and in the latter case, the set is included in the set of k for which $\frac{1}{2}n_K \leqslant |n_k| \leqslant n_K$. Moreover, if m is large enough, K satisfies (4.3). Hence, in either case it follows from (4.14) that

$$\begin{aligned} \sum_{2^m < |n_k| \leqslant 2^{m+1}} \left\{ |C_{n_k}|^2 \right\} &\leqslant \sum_{n_K/2 \leqslant |n_k| \leqslant n_K} \left\{ |C_{n_k}|^2 \right\} = O\left(n_K^{-1/q-\alpha p/p'} \right) \\ &= O\left(2^{m(-1/q-\alpha p/p')} \right). \end{aligned}$$

Also, by (4.8), the number of terms in the above summation is $O(2^m)$. Hence, by Cauchy's inequality,

$$\begin{aligned} \sum_{2^m < |n_k| \leqslant 2^{m+1}} |C_{n_k}| &\leqslant \left\{ \sum_{2^m < |n_k| \leqslant 2^{m+1}} |C_{n_k}|^2 \right\}^{1/2} \left\{ \sum_{2^m < |n_k| \leqslant 2^{m+1}} 1 \right\}^{1/2} \\ &= O\left\{ 2^{m/2((1-\alpha p)/p')} \right\}, \end{aligned}$$

so

$$\sum_{k=-\infty}^{\infty} |C_{n_k}| \leqslant C \sum_{m=1}^{\infty} 2^{m(1-\alpha p)/2p'}. \tag{4.15}$$

Since $\alpha p > 1$, the series on the right side is convergent. This completes the proof of Theorem 1.

Theorems 2 and 3 can be proved in a similar manner.

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