A DOUBLE WEIGHT EXTRAPOLATION THEOREM

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Abstract. If an operator is of weak type \((p_0, p_0)\) with weights \((u, v)\) for every \((u, v) \in A_{p_0}\), then the same holds for \(1 < p < p_0\).

1. A nonnegative function \(u\) on \(\mathbb{R}^n\) is said to be in \(A_p\) iff \((\int_Q u) \cdot (\int_Q u^{1-p'})^{p-1} \leq c|Q|^p\), and a pair of nonnegative functions \((u, v)\) is in \(A_p\) iff \((\int_Q u) \cdot (\int_Q v^{1-p'})^{p-1} \leq c|Q|^p\). The smallest \(c\) for which these inequalities hold for all cubes \(Q \subset \mathbb{R}^n\) will be referred to as the \(A_p\)-constant of \(u\), \((u, v)\), respectively. These classes were introduced by Muckenhoupt [3] and are important in the study of weighted norm inequalities for the Hardy-Littlewood maximal operator

\[ Mf(x) = \sup \frac{1}{|Q|} \int_Q |f|, \]

the Hilbert transform, and many others. In [1] we find the following remarkable theorem.

Theorem 1. Let \(T\) be a sublinear operator, \(1 \leq p_0 < \infty\), and

\[ u\{x: |Tf(x)| > y\} \leq \frac{B}{y^{p_0}} \int |f|^{p_0} u \quad \text{for every } u \in A_{p_0}, \]

where \(B\) depends only on the \(A_{p_0}\)-constant of \(u\). Then, if \(1 < p < \infty\) and \(u \in A_p\),

\[ u\{x: |Tf(x)| > y\} \leq \frac{C}{y^p} \int |f|^p u, \]

where \(C\) depends only on the \(A_p\)-constant of \(u\).

This theorem remains true if all weak type inequalities are replaced by strong type inequalities, and in this setting was first proved by Rubio de Francia [5]. The method of the proof of Theorem 1 given by Garcia-Cuerva in [1] is different and, as we shall see, can be modified to obtain a double weight \((u, v)\) version of Theorem 1.

2. The results which we obtain are best stated in the terminology of \(L(p, q, \lambda)\)-spaces [2]. If \(\lambda\) is a Borel measure on \(\mathbb{R}^n\) and, for \(f: \mathbb{R}^n \to \mathbb{R}\),

\[ f_{\lambda}^*(t) = \inf\{ y: \lambda\{|f| > y\} \leq t \}, \]

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the nonincreasing rearrangement of \( f \) with respect to the measure \( \lambda \), then we write

\[
\|f\|_{p,q;\lambda} = \left( \int_0^\infty \left[ t^{1/p} f^{*}(t) \right]^q \frac{dt}{t} \right)^{1/q}, \quad \text{if } 1 \leq p, q < \infty,
\]

and \( \|f\|_{p;\infty;\lambda} = \sup_{t>0} t^{1/p} f^{*}(t), \ 1 \leq p < \infty. \) If \( d\lambda = udx \), we write \( \|f\|_{p,q;u} = \|f\|_{p,q;\lambda} \). The weak type inequalities in Theorem 1 can then be written as \( \|Tf\|_{p,\infty;u} \leq C\|f\|_{p,u} \).

The method in [1] carries immediately over to prove the following

**Theorem 2.** Let \( T \) be a sublinear operator, \( 1 \leq p_0 < \infty, \) and \( \|Tf\|_{p_0,\infty;u} \leq B\|f\|_{p_0,u} \) for every \( (u,v) \in A_{p_0} \), where \( B \) depends only on the \( A_{p_0} \)-constant of \( (u,v) \). Let \( 1 < p < \infty, \) and let \( (u,v) \in A_p \) such that \( \|Mf\|_{p',\infty} \leq C\|f\|_{p,u} \) and \( \|Mf\|_{p,u} \leq C\|f\|_{p,v} \). Then \( \|Tf\|_{p,\infty;u} \leq C\|f\|_{p,v} \) where \( C \) depends only on the \( A_p \)-constant of \((u,v)\).

The proof is exactly the same as the one given in [1] except now \( G = \{M(\gamma^{1/p}u)_{v^{-1}}\} \). The strong type norm inequalities on the maximal function make Lemmas 1, 2 of [1] valid statements, i.e.,

**Lemma 1.** Let \( (u,v) \in A_p \) for some \( 1 < p < \infty, \) and for \( 0 < t < 1 \) and \( g \geq 0 \), let \( G = \{M(\gamma^{1/p}u)_{v^{-1}}\} \).

(i) If \( p_0 = p - tp'/p' \), then \( (gu,Gv) \in A_{p_0} \) with \( A_{p_0} \)-constant no larger than the \( A_p \)-constant of \((u,v)\) raised to the \( 1 - t \) power.

(ii) If \( \|Mf\|_{p',\infty} \leq C\|f\|_{p,u} \), then \( \|G\|_{p',u} \leq C\|g\|_{p',u} \).

3. It is well known that \( (u,v) \in A_p \) if and only if \( (v^{1-p'}, u^{1-p'})' \in A_{p'} \), and that these are equivalent with \( \|Mf\|_{p,\infty;u} \leq B\|f\|_{p,v} \). However, strong type inequalities for \( M \) need not hold (see [4]), and thus the strong type inequalities for \( M \) in the hypothesis of Theorem 2 are unnatural. We will prove the following

**Theorem 3.** Let \( 1 < p_0 < \infty, \) and let \( T \) be a sublinear operator such that \( \|Tf\|_{p_0,\infty;u} \leq B\|f\|_{p_0,u} \) for every \( (u,v) \in A_{p_0} \), where \( B \) depends only upon the \( A_{p_0} \)-constant of \((u,v)\). If \( 1 < p < p_0 \) and \( (u,v) \in A_p \), then \( \|Tf\|_{p,\infty;u} \leq C\|f\|_{p,v} \), where \( C \) depends only upon the \( A_p \)-constant of \((u,v)\).

The proof is based on the following

**Lemma 2.** Let \( 1 < p < p_0 < \infty, \) \( (u,v) \in A_p \), and \( g \geq 0 \) in \( L^{p/(p_0-p)}(v) \). Then there exists a function \( G \geq 0 \) such that

(i) \( u(G > y) \leq \frac{c}{y^{p/(p_0-p)}} \int g^{p/(p_0-p)}v, \) and

(ii) \( (G^{-1}u, g^{-1}v) \in A_{p_0} \), where the constants involved depend only on the \( A_p \)-constant of \((u,v)\).

**Proof.** We note that \( p_0' < p' \) and \( (v^{1-p'}, u^{1-p'})' \in A_{p'} \). We set

\[
t = \left( p' - p_0' \right) / \left( p' - 1 \right)
\]

so that \( p_0' = p' - tp'/p \) and \( (p/t)' = p'/p_0' \). We define \( h \) by the relation

\[
g^{p/(p_0-p)}v = h^{(p_0-1)/(p_0-p)}v^{1-p'},
\]
and we set (as in Lemma 1)

\[ H = \{ M(h^{1/p^\prime}v^{1-p^\prime})u^{p^\prime-1}\}^t. \]

Since \((u, v) \in A_p\), we get

\[ u\{ M(h^{1/p^\prime}v^{1-p^\prime}) > y^{1/t}\} \leq \frac{c}{y^{p^\prime/t}} \int h^{p^\prime}v^{1-p^\prime}, \]

and hence

\[ (1) \quad u\{ Hu^{(1-p^\prime)} > y\} \leq \frac{c}{y^{p^\prime/(p_0-p)}} \int g^{p^\prime/(p_0-p)}v. \]

We now let \( G = H^{p_0-1}u^{-(p_0-p)/(p-1)} \), and using (1) we check that

\[ u\{ G > y\} \leq \frac{c}{y^{p^\prime/(p_0-p)}} \int g^{p^\prime/(p_0-p)}v. \]

Since, by Lemma 1, \((hu^{1-p'}, Hu^{1-p'}) \in A_{p_0}^0\), we see that

\[ \left( H^{1-p_0}u^{(1-p')v(1-p_0)}, H^{1-p_0}v^{(1-p')(1-p_0)} \right) \in A_{p_0}. \]

The proof is completed by noting that \( G^{-1}u = H^{1-p_0}u^{(p_0-1)(p-1)}, \) and \( g^{-1}v = h^{1-p_0}v^{(1-p')(1-p_0)}. \)

**Proof of Theorem 3.** Let \( E_s = \{ x: |Tf(x)| > s \}. \) Note that \( \|f\|_{p_0} = \|f\|_{p_0,v} = \int f|g^{p_0/(p_0-p)}v| \) for some \( \int g^{p_0/(p_0-p)}v = 1. \) If \( r = p/p_0, \) then

\[ s^{p_0}u(E_s)^{p_0/p} = s^{p_0}\left( \int X_{E_s} G \cdot G^{-1}u \right)^{1/r} \leq s^{p_0}\|X_{E_s}\|^{1/r}_{l^1;G^{-1}u} \cdot \|G\|^{1/r}_{l^1;G^{-1}u}, \]

where \( \sigma = 1/r. \)

We first estimate \( \|G\|_{\sigma^\prime,\infty;G^{-1}u}. \) The distribution function of \( G \) with respect to the measure \( G^{-1}udx \) is

\[ G^{-1}u \{ G(x) > y \} = \int_{\{ G(x) > y \}} G^{-1}u \leq \frac{1}{y} \int_{\{ G(x) > y \}} u \leq \frac{c}{y^{p_0/(p_0-p)}} \]

by (i) of Lemma 2. Hence

\[ (G)^{G^{-1}u}(t) \leq \frac{c}{t^{(p_0-p)/p_0}} = \frac{c}{t^{1/\sigma^\prime}}, \]

and so

\[ \|G\|_{\sigma^\prime,\infty;G^{-1}u} = \sup_{t > 0} \left\{ t^{1/\sigma'}(G)^{G^{-1}u}(t) \right\} \leq c. \]

For the estimation of

\[ \|X_{E_s}\|_{\sigma;G^{-1}u} = \int_0^\infty t^{1/\sigma}(X_{E_s})^{G^{-1}u}(t) \frac{dt}{t}, \]

we note that

\[ G^{-1}u \{ X_{E_s} > y \} = G^{-1}u(E_s) \cdot \chi_{[0,1]}(y) \leq \frac{c}{s^{p_0}} \|f\|_{p_0,v} \cdot \chi_{[0,1]}(y). \]
From this we obtain that
\[ (X^*_{E_t}, G^{-1}u(t) \leq X_{[0,R]}(t), \quad \text{where } R = \frac{c}{s^p_0} \|f\|_{p,v}^p, \]
and, consequently,
\[ \|X_E\| \leq \int_0^R r^{1/\sigma-1} \, dt = cR^{1/\sigma} = c\left(\frac{1}{s^p_0} \|f\|_{p,v}^p\right). \]
This implies that
\[ s^{p_0}u(E_t)_{p_0/p} \leq c \frac{1}{s^p_0} \|f\|_{p,v}^p \quad \text{or} \quad s(u(E_t))_{1/p} \leq c \|f\|_{p,v}, \]
which was to be proved.

4. It is not known to the writer whether Theorem 3 is true if \( p > p_0 \). In order to formulate a substitute result for \( p > p_0 \) we rewrite
\[ \int |f|^{p} = \int |f|^{p}v^{1-p'} = \int \left(|f|^{|p'-1} \right)^{p}v^{1-p'} \]
\[ = \int_0^{\infty} \left(fv^{p'-1}\right)^*_{p',\sigma-1}(t) \, dt = \|fv^{p'-1}\|_{p',\sigma-1}^{p}. \]

**Theorem 4.** Let \( 1 < p_0 < p < \infty \), and let \( T \) be a sublinear operator so that, for every \( (u, v) \in A_{p_0} \),
\[ \|Tf\|_{p,\infty,u} \leq c \|f\|_{p_0,v} = c \|fv^{p_0-1}\|_{p_0,p_0;\sigma,\sigma}^{p_0-1}, \]
where \( c \) only depends on the \( Ap_0 \)-constant of \( (u, v) \). Then, if \( (u, v) \in A_p \),
\[ \|Tf\|_{p,\infty,u} \leq c \|fv^{p_0-1}\|_{p,p_0;\sigma,\sigma}^{p_0-1}, \]
where \( c \) only depends upon the \( Ap \)-constant of \( (u, v) \).

**Remark.** Since \( \|fv^{p_0-1}\|_{p,p_0;\sigma,\sigma} \geq \|fv^{p_0-1}\|_{p,p_0;\sigma,\sigma} = \|f\|_{p,v} \) (see [2]), the conclusion of Theorem 4 is not as strong as the corresponding weak type inequality.

The proof is based on a slight reformulation of Lemma 1, which we state as

**Lemma 3.** Let \( 1 < p_0 < p < \infty \), \( (u, v) \in A_{p_0} \), and let \( t = (p-p_0)/(p-1) \). If \( g \geq 0 \) is in \( L^{p/(p-p_0)}(u) \), and \( G = \{ M(g^{1/t}u)^{1/t} \} \), then
\[ (i) \quad v^{1-p'}\{ M(\{ g^{1/t}u \}^{1/t}) > y \} \leq \frac{c}{y^{p/(p-p_0)}} \int g^{p/(p-p_0)}u, \quad \text{and} \]
\[ (ii) \quad (gu, Gv) \in A_{p_0}. \]
The constants in (i) and (ii) depend only on the \( Ap_0 \)-constant of \( (u, v) \).

**Proof.** Since \( (v^{1-p'}, u^{1-p'}) \in A_p \) with the same \( Ap \)-constant as the \( Ap_0 \)-constant for \( (u, v) \), we see that
\[ \int_{\{ M(g^{1/t}u)^{1/t} > y^{1/t} \}} v^{1-p'} \leq \frac{c}{y^{p/(p-p_0)}} \|g\|_{p/(p-p_0),u}^{p/(p-p_0)}, \]
and this is (i). The proof of (ii) is exactly the same as in [1].
PROOF OF THEOREM 4. Let $E_s = \{ x : |Tf(x)| > s \}$ and note that $s^{p_0} u(E_s)^{p_0/p} = s^{p_0} \|X_{E_s}\|_{p/p_0,u} = s^{p_0} \|X_{E_s}\|_{u,p_0} = \|g\|_{(p/p_0)'} u = 1$. By (ii) of Lemma 3,

$$s^{p_0} u(E_s)^{p_0/p} \leq c \int |f|^{p_0} G u = c \int |f|^{p_0} v^{p'-1} M \left(g^{1/t} u\right)^t v^{-p'} \leq c \left\|f v^{p'-1}\right\|_{p/p_0,t;1} \left\|M \left(g^{1/t} u\right)^t v^{-p'}\right\|_{p'/1,\infty;v^{-p'}} \leq c \left\|f v^{p'-1}\right\|_{p_0,t;1}^{p_0},$$

since $(p'/t)' = p/p_0$. This completes the proof.

REFERENCES


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