EVERY UNIFORMLY CONTINUOUS CENTERED SEMIGROUP IS NORMAL

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Abstract. A well-known result on commutators of operators is used to prove the assertion in the title.

Consider a strongly continuous semigroup \((S_t; t \geq 0)\) of continuous linear operators on a Hilbert space \(\mathcal{H}\). Let \(U_tP_t\) be the canonical polar factorization of \(S_t\). The semigroup \((S_t)\) is said to be centered if the partially isometric factor \((U_t)\) of \((S_t)\) is a semigroup. Examples of strongly continuous nonnormal centered semigroups are given in [1]. The purpose of this note is to demonstrate the truth of the assertion given in the title.

Theorem. Every uniformly continuous centered semigroup is normal.

Proof. Let \((S_t)\) be a uniformly continuous centered semigroup with (bounded) generator \(A\); that is, \(S_t = e^{tA}\). We need only show that the operator \(A\) is normal and we shall do so. Consider the power series expansions of \(S_t*S_t\) and \(S_tS_t^*\):

\[
S_t*S_t = I + t(A + A^*) + t^2(A^2 + 2AA^* + A^{*2}) + O(t^3)
\]

and

\[
S_tS_t^* = I + t(A + A^*) + t^2(A^2 + 2AA^* + A^{*2}) + O(t^3).
\]

Since \((S_t)\) is centered, \(S_t*S_t\) commutes with \(S_tS_t^*\) for all nonnegative \(t\) and \(r\) [1, Lemma 2]. Consequently, the coefficients in the power series expansions of \(S_t*S_t\) and \(S_tS_t^*\) form an abelian set of operators. In particular, \(A + A^*\) commutes with \(A^2 + 2AA^* + A^{*2}\) and \(A^2 + 2AA^* + A^{*2}\), and with their difference \(2(A^*A - AA^*)\). That \(A\) is normal follows from the next lemma.

Lemma. An operator \(A\) on a Hilbert space \(\mathcal{H}\) is normal if \(A + A^*\) commutes with \(A^*A - AA^*\).

Proof. Write \(A = H + iK\) where \(H = (A + A^*)/2\) and \(K = (A - A^*)/2i\). If \(A + A^*\) commutes with \(A^*A - AA^*\), then \(H\) commutes with \(HK = KH\). Kleinecke’s result [2] on commutators implies that 0 is the only element in the spectrum \(HK - KH\). Since \(H\) and \(K\) are Hermitian, this in turn implies that \(HK - KH = 0\) or equivalently that \(A\) is normal.

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Using the Theorem one can easily construct examples of semigroups for which the generator has nice algebraic properties not inherited by the semigroup itself. For example, if $A$ is a nonnormal centered operator [3] (that is, \{ $A^m A^n, A^n A^m$: $m, n$ nonnegative integers $\}$ is abelian), then $(e^{tA})$ is nonnormal and, consequently, not centered. Similarly, if $A$ is a nonnormal, subnormal operator, then $(e^{tA})$ is not centered.

We showed in the proof of the Theorem that $A$ is normal whenever the coefficients of the power series expansion of $S_i S_i^*$ commute with those of $S_i S_i^*$. These coefficients also arise in [4] where Lambert shows that $A$ is subnormal if each of the coefficients of $S_i S_i^*$ is nonnegative definite.

REFERENCES


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