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# UNORIENTED BRANCHED COVERINGS ARISING FROM GROUP ACTIONS

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ABSTRACT. For an unbranched covering $f\colon M^n \to N^n$, $[M] = (\deg f)[N]$ in unoriented cobordism $\mathfrak{N}_*$. Thus, in general, if $f\colon M \to N$ is a branched covering, then $[M] - (\deg f)[N]$ depends upon the branching behavior.

In this note we describe the ideal $I_G$ of unoriented cobordism classes $[M^n] - |G|[M^n/G]$, where $G$ is a finite group acting on $M$ so that $M \to M/G$ is a $|G|$-fold smooth branched covering of closed smooth manifolds.

**1. Introduction.** The purpose of this note is to describe the ideal $I_G$ of unoriented cobordism classes $[M^n] - |G|[M^n/G]$, where $G$ is a finite group acting on $M$ so that $M \to M/G$ is a $|G|$-fold smooth branched covering of closed smooth manifolds.

It is known that for an unbranched covering $f\colon M^n \to N^n$, $[M] = (\deg f)[N]$ in unoriented cobordism $\mathfrak{N}_*$. Thus, in general, if $f\colon M \to N$ is a branched covering, then $[M] - (\deg f)[N]$ depends upon the branching behavior.

We follow Brand [**1**] in defining branched covering, including a smoothness condition:

DEFINITION. A *branched covering* is a smooth map $f\colon M^n \to N^n$ between closed smooth manifolds which is finite-to-one and open. The *singular set* is the set of points of $M$ at which $F$ is not a local homeomorphism, and the *branch set* $B_f$ is the image under $f$ of the singular set. We assume that $B_f$ is a smoothly embedded codimension 2 submanifold of $N$.

Let $B_k^{n-2}$ be a component of $f^{-1}B_f$. Then, on a neighborhood of $B_k$, $f$ is locally isomorphic to the map $D^{n-2} \times D^2 \to D^{n-2} \times D^2$: $(x, z) \to (x, z^k)$. The number $k$ is called the *local branching degree*. For more details, see Stong [**5**].

Note that $M - f^{-1}B_f \to N - B_f$ is an unbranched covering. If $M$ and $N$ are path-connected and if $\pi_1(N - B_f)/\pi_1(M - f^{-1}B_f) \simeq G$, a finite group with $|G| = \deg f$, then $G$ acts on $M$, freely on $M - f^{-1}B_f$, with cyclic isotropy groups, and $M \to M/G = N$ is the branched covering.

Conversely, if $G$ acts on $M$ with cyclic isotropy groups such that $\{x \in M\colon G_x \neq \{1\}\}$ is a codimension 2 submanifold of $M$, then $M \to M/G$ is a branched covering. If $F_k$ is a component of the fixed point set of $\mathbf{Z}_k \subset G$, then $k$ is the local branching degree on $F_k$.

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NOTATION. If $G = \mathbf{Z}_d$, then we write $I_d$ for $I_G$. We write $\mathbf{R}P(n_1,\ldots,n_k)$ to mean the real projective space of the bundle $\gamma_{n_1} \oplus \cdots \oplus \gamma_{n_k}$ sitting over $\mathbf{R}P^{n_1} \times \cdots \times \mathbf{R}P^{n_k}$, where $\gamma_{n_i}$ is the standard line bundle over $\mathbf{R}P^{n_i}$ [**6**].

In §2, we determine $I_G$ in unoriented cobordism $\mathfrak{N}_*$ for all finite groups $G$. We begin by noting the following reduction of the problem:

PROPOSITION A. *If $G$ is a finite group with a Sylow 2-subgroup $S$, then $I_G = I_S$ in $\mathfrak{N}_*$.*

Stong [**6**] shows that $I_2 = \oplus_n\{\alpha \in \mathfrak{N}_n\colon w_1^n(\alpha) = 0\}$ in $\mathfrak{N}_*$.

PROPOSITION B. *For $s > 1$, $I_{2^s} = \oplus_n\{\alpha \in \mathfrak{N}_n\colon w_1^i w_{n-i}(\alpha) = 0, 1 \leqslant i \leqslant n\}$ in $\mathfrak{N}_*$.*

PROPOSITION C. *If $S$ is a 2-group containing $\mathbf{Z}_2 \times \mathbf{Z}_2$, then $I_S = 0$ in $\mathfrak{N}_*$.*

If $S$ is a nonabelian 2-group that does not contain $\mathbf{Z}_2 \times \mathbf{Z}_2$, then $S$ is a generalized quaternion group. Therefore, we finish our study with the following:

PROPOSITION D. *If $Q$ is a generalized quaternion group, then $I_Q$ in $\mathfrak{N}_*$ is the ideal generated by $[\mathbf{R}P(0, m, 2j+1, 2j+1)]$, $m \geqslant 4j+3$.*

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**2. Unoriented branched coverings.** Many of our results rely upon the following proposition due to Stong [**5**].

PROPOSITION 2.1. *If $f\colon M^n \to N^n$ is a branched covering, the $[M] - (\deg f)[N] = [\mathbf{R}P(\nu_{\text{even}} \oplus 1)]$ in $\mathfrak{N}_*$, where $\nu_{\text{even}} = \bigcup_{k\,\text{even}} \nu_k$, with $\nu_k$ the normal bundle of $B_k$ in $M$.* □

The following reduction of the problem follows from the above result.

PROPOSITION 2.2. *If $G$ is a finite group with a Sylow 2-subgroup $S$, then $I_G = I_S$ in $\mathfrak{N}_*$.*

PROOF. If $M^n \to M^n/G$ is a branched covering, then

$$\begin{aligned}[M] - |G|[M/G] &= [M] - |S|[M/S] + |S|\{[M/S] - [G:S][M/G]\}\\ &= [M] - |S|[M/S],\end{aligned}$$

and $M \to M/S$ is a branched covering. Thus, $I_G \subset I_S$. (Note: By Proposition 2.1, if $S = \{1\}$, then $[M] - |G|[M/G]$ is 0 because $M \to M/G$ is a branched covering with only odd local branching degrees.)

If $M \to M/S$ is a branched covering, then $G$ acts on $G \times_S M$ as multiplication on $G$. $G \times_S M \to G \times_S M/G \simeq M/S$ is a branched covering with $[G \times_S M] - |G|[M/S] = [G:S][M] - |G|[M/S] = [G:S]\{[M] - |S|[M/S]\} = [M] - |S|[M/S]$. Thus $I_S \subset I_G$. □

OBSERVATION. According to Stong [**6**], $I_2 = \{\alpha\colon w_1^n(\alpha) = 0\}$ in $\mathfrak{N}_*$.

PROPOSITION 2.3. *For $s > 1$, $I_{2^s} = \{\alpha\colon w_1^i w_{n-i}(\alpha) = 0, 0 \leqslant i \leqslant n\}$ in $\mathfrak{N}_*$.*

PROOF. Let $M^n \to M^n/\mathbf{Z}_{2^s}$ be a branched covering, and let $F^{n-2} \subset M^n$ be the singular set. $F = A \cup B$, where $A$ is the set of points with isotropy group $\mathbf{Z}_2$ and $B = F - A$.

Let $\nu \to F$ be the normal bundle of $F$ in $M$, and let $c \in H^1(\mathbf{R}P(\nu + 1); \mathbf{Z}_2)$ be the first Stiefel-Whitney class of the double cover by the sphere bundle. We will often consider $c$ to be the corresponding class in $H^1(\mathbf{R}P(\nu); \mathbf{Z}_2)$ because the classifying map $\mathbf{R}P(\nu) \to \mathbf{R}P^\infty$ factors through the inclusion and classifying map $\mathbf{R}P(\nu) \to \mathbf{R}P(\nu + 1) \to \mathbf{R}P^\infty$. If $w(\nu) = 1 + u_1 + u_2$, then $w(\mathbf{R}P(\nu)) = w(F)(1 + c)^2 + u_1(1 + c) + u_2 = w(F)(1 + u_1)$ since $c^2 + cu_1 + u_2 = 0$. Because $\nu|_B \to B$ is a complex vector bundle, $u_1|_B = 0$, implying that $w(\mathbf{R}P(\nu)) = w(A)(1 + u_1) + w(B)$.

$\mathbf{Z}_{2^s}/\mathbf{Z}_2 \simeq \mathbf{Z}_{2^{s-1}} \neq \{1\}$ acts freely on $A$. $\mathbf{Z}_2$ acts as multiplication by $-1$ in the fibers of $\nu|_A \to A$; thus, $\mathbf{Z}_2$ acts as $\det(-I_2) = 1$ on $\det \nu|_A \to A$. It follows that $\mathbf{Z}_{2^{s-1}}$ acts freely on $\det \nu|_A \to A$; hence, $\det \nu|_A \to A$ bounds. Because $w(\det \nu|_A) = 1 + u_1$, $u_1^i w_\omega(A)[A] = 0$. In particular, $0 = \{w(A)(1 + u_1)\}_\omega[A] = w_\omega(B)[B]$ since according to Stong [5], $\mathbf{R}P(\nu) \to \mathbf{R}P^\infty$ bounds.

$$\begin{aligned} w(\mathbf{R}P(\nu + 1)) &= w(F)\{(1 + c)^3 + u_1(1 + c)^2 + u_2(1 + c)\} \\ &= w(F)\{1 + (c + u_1) + (c^2 + u_2)\}, \end{aligned}$$

since $c^3 + u_1c^2 + u_2c = 0$. Recall that $[\mathbf{R}P(\nu + 1)] = [M]$ in $\mathfrak{N}_*$.

$$w_n[\mathbf{R}P(\nu + 1)] = w_{n-2}(F)(c^2 + u_2)[\mathbf{R}P(\nu + 1)] = w_{n-2}(F)[F] = 0,$$

$$\begin{aligned} w_1w_{n-1}[\mathbf{R}P(\nu + 1)] &= (w_1(F) + c + u_1)(w_{n-2}(F)(c + u_1) \\ &\qquad + w_{n-3}(F)(c^2 + u_2))[\mathbf{R}P(\nu + 1)] \\ &= (2u_1w_{n-3}(F) + w_{n-2}(F) + w_{n-3}(F)w_1(F))[F] = 0. \end{aligned}$$

We know that $w_1^n[M] = 0$, so assume $2 \leqslant i \leqslant n - 2$.

$$\begin{aligned} w_1^iw_{n-i}(\mathbf{R}P(\nu + 1)) &= (w_1(F) + c + u_1)^i(w_{n-i}(F) + w_{n-i-1}(F)(c + u_1) \\ &\qquad + w_{n-i-2}(F)(c^2 + u_2)) \\ &= \sum_{j=0}^{i} \binom{i}{j} c^j(w_1(F) + u_1)^{i-j}(w_{n-1}(F) + w_{n-i-1}(F)u_1 + w_{n-i-2}(F)u_2) \\ &\quad + \sum_{j=0}^{i} \binom{i}{j} c^j(w_1(F) + u_1)^{i-j}(w_{n-i-1}(F)c + w_{n-i-2}(F)c^2). \end{aligned}$$

Now,

$$\begin{aligned} &c^j(w_1(F) + u_1)^{i-j}(w_{n-i}(F) + w_{n-i-1}(F)u_1)[\mathbf{R}P(\nu + 1)] \\ &\qquad = \bar{u}_{j-2}w_1(\mathbf{R}P(\nu))^{i-j}w_{n-i}(\mathbf{R}P(\nu))[F] = 0, \end{aligned}$$

where $\bar{u} = 1/(1 + u_1 + u_2)$.

$$
\begin{aligned}
&w_1^i w_{n-i}[\mathbf{R}P(\nu+1)] \\
&\quad = \sum_{j=2}^{i} \binom{i}{j} c^{j-1}(w_1(F)+u_1)^{i-j} w_{n-i-2}(F)(c^3+c^2u_1)[\mathbf{R}P(\nu+1)] \\
&\qquad + \sum_{j=0}^{i} \binom{i}{j} c^{j}(w_1(F)+u_1)^{i-j}(w_{n-i-1}(F)c + w_{n-i-2}(F)c^2)[\mathbf{R}P(\nu+1)] \\
&\quad = \sum_{j=2}^{i} \binom{i}{j} c^{j+1}(w_1(F)+u_1)^{i-j}(w_{n-i-2}(F)u_1 + w_{n-i-1}(F))[\mathbf{R}P(\nu+1)] \\
&\qquad + \sum_{j=0}^{1} \binom{i}{j} c^{j}(w_1(F)+u_1)^{i-j}(w_{n-i-1}(F)c + w_{n-i-2}(F)c^2)[\mathbf{R}P(\nu+1)] \\
&\quad = \left\{(w_1(F)+u_1)^i w_{n-i-2}(F) + i(w_1(F)+u_1)^{i-1} w_{n-i-1}(\mathbf{R}P(\nu))\right\}[F] \\
&\quad = (w_1(A)+u_1)^i w_{n-i-2}(A)[A] + w_1(B)^i w_{n-i-2}(B)[B] = 0.
\end{aligned}
$$

Thus, $w_1^i w_{n-i}[M] = 0$ for $0 \leqslant i \leqslant n$.

Set $J = \{\alpha\colon w_1^i w_{n-i}(\alpha) = 0,\ 0 \leqslant i \leqslant n\}$. According to Capobianco [**3**], $\alpha \in J$ can be represented by a fibering with fiber $\mathbf{R}P(3)$, and Stong [**6**] exhibits generators of $J$; they are of the form $\mathbf{R}P(\xi+\eta)$, where $\xi$ is a complex line bundle and $\eta$ is a real 2-plane bundle. There is a $\mathbf{Z}_{2^s}$ action on $\mathbf{R}P(\xi+\eta)$ induced by multiplication by the $2^s$th roots of 1 in the fibers of $\xi$, and by 1 in the fibers of $\eta$. The fixed set is $\mathbf{R}P(\xi) \cup \mathbf{R}P(\eta)$, which has codimension 2; thus, $J = I_{2^s}$. □

PROPOSITION 2.4. *If $S$ is a 2-group containing $\mathbf{Z}_2 \times \mathbf{Z}_2$, then $I_S = 0$ in $\mathfrak{N}_*$.*

PROOF. If $M \to M/S$ is a branched covering, then the isotropy groups of the $S$ action on $M$ are cyclic. In particular, $\mathbf{Z}_2 \times \mathbf{Z}_2$ acts without stationary points, implying $[M] = 0$ in $\mathfrak{N}_*$ [**4**, 30.1]. □

REMARK. If $S$ is a nonabelian 2-group that does not contain $\mathbf{Z}_2 \times \mathbf{Z}_2$, then $S$ is a generalized quaternion group.

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm ij\}$ be the quaternion group, let $M^n \to M^n/Q_8$ be a branched covering and let $F^{n-2}$ be the fixed point set. Then, $F = A_2 \cup B_i \cup C_j \cup D_{ij}$, where $A_2$ is the fixed point set of $\mathbf{Z}_2$ in $Q_8$, $B_i$ is the fixed point set of $\langle i \rangle$ in $Q_8$, etc.

$\langle j \rangle$ gives us a $\mathbf{Z}_4$ action on the normal bundle of $B_i$, $\nu \to B$, such that $\mathbf{Z}_2 \subset \mathbf{Z}_4$ acts trivially on $B$ and as multiplication by $-1$ in the fibers of $\nu$. Also, $\mathbf{Z}_4/\mathbf{Z}_2 \simeq \mathbf{Z}_2$ acts freely on $B$.

Beem [**2**] supplies us with a classifying space for such bundles: Write $\mathbf{C}^\infty = \mathbf{R}^\infty + j\mathbf{R}^\infty$. Then $j$ acts on $\mathbf{C}^\infty$ by multiplication. There is an induced action on $\gamma^2 \to BO_2(\mathbf{C}^\infty)$ such that $\mathbf{Z}_2 \subset \langle j \rangle$ acts trivially on $BO(\mathbf{C}^\infty)$ and as multiplication by $-1$ in the fibers of $\gamma^2$. We can classify a free $\mathbf{Z}_2$ action by a map into $[S^\infty, -1]$. Thus, the desired classifying space is $BO_2(\mathbf{C}^\infty) \times S^\infty/\mathbf{Z}_2$. Beem [**2**] determines that

$$H^*(BO_2(\mathbf{C}^\infty) \times S^\infty/\mathbf{Z}_2; \mathbf{Z}_2) = \mathbf{Z}_2[w_1, w_2^2, \alpha]/\langle \alpha^2 w_1 \rangle,$$

where $w_i, w_2$ are the universal classes which generate $H^*(BO_2; \mathbf{Z}_2)$ and $\alpha$ generates $H^*(\mathbf{R}P^\infty; \mathbf{Z}_2)$.

We have the following commutative diagram:

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & BO_2(\mathbf{C}^\infty) \times S^\infty \arrow[d] \\
B/\mathbf{Z}_2 \arrow[r] & BO_2(\mathbf{C}^\infty) \times S^\infty/\mathbf{Z}_2
\end{tikzcd}$$

The classes $w_1$ and $w_2^2$ pull back to $u_1$ and $u_2^2$ in $H^*(B; \mathbf{Z}_2)$, where $w(\nu) = 1 + u_1 + u_2$. Note that $u_1 = 0$ because $\nu \to B$ is a complex vector bundle. Thus, $w_\omega u_2^{2j}[B] = 0$ for all $\omega$, $j$ since these numbers come from $B/\mathbf{Z}_2$. We obtain the same result for $A$, $C$, and $D$.

In fact, for $A$ we can say more. There is a $Q_8$ action on $\gamma^2 \to BO_2(\mathbf{H}^\infty)$ induced by multiplication such that $\mathbf{Z}_2 \subset Q_8$ acts trivially on $BO_2(\mathbf{H}^\infty)$ and as multiplication by $-1$ in the fibers of $\gamma^2$. We have such an action on $\nu \to A$; also, $Q_8/\mathbf{Z}_2 \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$ acts freely on $A$. Such an action can be classified by a map into $[S^\infty \times S^\infty; -1 \times 1, 1 \times -1]$. Thus, we have a classifying space for $A/\mathbf{Z}_2 \times \mathbf{Z}_2$, namely $BO_2(\mathbf{H}^\infty) \times S^\infty \times S^\infty/i \times -1 \times 1,\ j \times 1 \times -1$, and $H^*(BO_2(\mathbf{H}^\infty) \times S^\infty \times S^\infty/\mathbf{Z}_2 \times \mathbf{Z}_2; \mathbf{Z}_2) \simeq \mathbf{Z}_2[\alpha, \beta, w_1, w_2^4]/\text{relations}$.

From the double covers $A \to A/\mathbf{Z}_2$ and $BO_2(\mathbf{C}^\infty) \times S^\infty \to BO_2(\mathbf{C}^\infty) \times S^\infty/\mathbf{Z}_2$, we have the following exact sequences:

$$\begin{tikzcd}
H^*(A; \mathbf{Z}_2) \arrow[rr, bend left, "\delta"] & H^*(A/\mathbf{Z}_2; \mathbf{Z}_2) \arrow[l] & H^*(A \times I/\mathbf{Z}_2 \times -1; \mathbf{Z}_2) \arrow[l] \\
H^*(BO_2 \times S^\infty; \mathbf{Z}_2) \arrow[u] \arrow[rr, bend right] & H^*(BO_2 \times S^\infty/\mathbf{Z}_2; \mathbf{Z}_2) \arrow[l] \arrow[u] \arrow[d, equal] & H^*(BO_2 \times S^\infty \times 1/\mathbf{Z}_2 \times -1; \mathbf{Z}_2) \arrow[l] \arrow[u] \\
\mathbf{Z}_2[w_1, w_2] \arrow[rr, bend right] & \mathbf{Z}_2[w_1, w_2^2, \alpha]/\langle \alpha^2 w_1 \rangle \arrow[l] & (\mathbf{Z}_2[w_1, w_2^2, \alpha]/\langle \alpha^2 w_1 \rangle) \cdot U, \arrow[l, "\cdot\alpha"']
\end{tikzcd}$$

where $U$ is the Thom class of the line bundle $BO_2(\mathbf{C}^\infty) \times S^\infty \times 1/\mathbf{Z}_2 \times -1 \to BO_2(\mathbf{C}^\infty) \times S^\infty/\mathbf{Z}_2$.

$$\begin{aligned}
\langle w_\omega u_1^r u_2^{4j+1}, [A] \rangle &= \langle w_\omega u_1^r u_2^{4j+1}, \partial[A \times I/\mathbf{Z}_2 \times -1] \rangle \\
&= \langle \delta(w_\omega u_1^r u_2^{4j+1}), [A \times I, \mathbf{Z}_2 \times -1] \rangle \\
&= \langle w_\omega u_1^r u_2^{4j}(\delta u_2), [A \times I, \mathbf{Z}_2 \times -1] \rangle.
\end{aligned}$$

In the cohomology of the universal space, $\delta w_2$ is a class such that $\alpha(\delta w_2) = 0$; thus, $\delta w_2 = \alpha w_1 U$, implying

$$\begin{aligned}
\langle w_\omega u_1^r u_2^{4j+1}, [A] \rangle &= \langle w_\omega u_1^r u_2^{4j} \alpha u_1 U, [A \times I/\mathbf{Z}_2 \times -1] \rangle \\
&= \langle w_\omega u_1^{r+1} u_2^{4j} \alpha, [A/\mathbf{Z}_2] \rangle = 0
\end{aligned}$$

since $u_2^{4j}$ comes from $H^*(A/\mathbf{Z}_2 \times \mathbf{Z}_2; \mathbf{Z}_2)$.

We consider equivariant bordism of our special $Q_8$ actions, that is, $Q_8$ actions such that each action is free off of a codimension 2 submanifold, and on that

submanifold the isotropy groups are nontrivial cyclic groups. We obtain Conner-Floyd exact sequences:

$$\begin{tikzcd}
\mathfrak{N}_*(Q_8) \arrow[r] \arrow[d] & \mathfrak{N}_*(Q_8, \operatorname{Free}\partial) \arrow[r] \arrow[d, "\text{transfer}"] & \mathfrak{N}_{*-1}(\operatorname{Free} Q_8) \arrow[d] \arrow[dll, bend left] \\
\mathfrak{N}_*(\mathbf{Z}_2) \arrow[r] & \mathfrak{N}_*(\mathbf{Z}_2, \operatorname{Free}\partial) \arrow[r] & \mathfrak{N}_{*-1}(\operatorname{Free}\mathbf{Z}_2) \arrow[ull, bend left]
\end{tikzcd}$$

where the vertical arrows are obtained by restricting to the $\mathbf{Z}_2 \subset Q_8$ action. Classification of the normal bundle to the fixed set leads to

$$\begin{aligned}\mathfrak{N}_*(Q_8, \operatorname{Free}\partial) \simeq\ & \mathfrak{N}_{*-2}(BO_2(\mathbf{H}^\infty) \times S^\infty \times S^\infty/\mathbf{Z}_2 \times \mathbf{Z}_2) \\ & + 3\mathfrak{N}_{*-2}(BU_1(\mathbf{H}^\infty) \times S^\infty/\mathbf{Z}_2),\end{aligned}$$

and $\mathfrak{N}_*(\mathbf{Z}_2, \operatorname{Free}\partial) \simeq \mathfrak{N}_{*-2}(BO_2(\mathbf{H}^\infty))$. A class in $\mathfrak{N}_{*-2}(BO_2)$ is determined by the numbers $w_\omega u_1^r u_2^s[F]$; in fact, the numbers $w_\omega u_1^r u_2^s[A]$ determine a class in the image of transfer: Since $\mathbf{R}P(\nu) \to \mathbf{R}P^\infty$ bounds,

$$0 = c^{4j+3} w_\omega[\mathbf{R}P(\nu)] = \bar{u}_{4j+2}\{(1+u_1)w(A)\}_\omega[A] + u_2^{2j+1} w_\omega[B \cup C \cup D].$$

LEMMA 2.5. *The image of the transfer map*

$$\operatorname{tr}\colon \mathfrak{N}_*(BO_2(\mathbf{H}^\infty) \times S^\infty \times S^\infty/\mathbf{Z}_2 \times \mathbf{Z}_2) \to \mathfrak{N}_*(BO_2(\mathbf{H}^\infty))$$

*is a free $\mathfrak{N}_*$-module with a basis in the image of the homology transfer.*

PROOF. Let $\pi\colon \tilde{X} \to X$ be a double cover. There is a line bundle $l$ associated to this cover with $El = \tilde{X} \times \mathbf{R}/\mathbf{Z}_2 \times -1$. Let $[M, f] \in \mathfrak{N}_*(X)$; then, we have a double cover of $M$, $\tilde{M} = \{(m, \tilde{x}) \in M \times \tilde{X}\colon f(m) = \pi(\tilde{x})\}$. The Thom isomorphism $T$: $\mathfrak{N}_*(X) = \tilde{\mathfrak{N}}_{*+1}(Tl) \simeq \mathfrak{N}_{*+1}(Dl, Sl)$ sends $[M]$ to $[\tilde{M} \times I/\mathbf{Z}_2 \times -1, \tilde{M} \times \{\pm 1\}/\mathbf{Z}_2 \times -1]$, and the boundary homomorphism $\partial\colon \mathfrak{N}_{*+1}(Dl, Sl) \to \mathfrak{N}_*(Sl) \simeq \mathfrak{N}_*(\tilde{X})$ sends the given pair to $[\tilde{M}]$. The composite $\partial T\colon \mathfrak{N}_*(X) \to \mathfrak{N}_*(\tilde{X})$ is just the transfer map associated to the double cover $\tilde{X} \to X$.

If $f\colon Y \to Z$ is a map, then the image of $f_*\colon \mathfrak{N}_*(Y) \to \mathfrak{N}_*(Z)$ is a free $\mathfrak{N}_*$-module with a basis in the image of $f_*\colon H_*(Y) \to H_*(Z)$. Now $\partial$ is the composite of $i_*$: $\mathfrak{N}_{*+1}(Dl, Sl) \to \tilde{\mathfrak{N}}_{*+1}(\Sigma Sl)$, where $i$ is the inclusion, and the suspension isomorphism $\sigma\colon \tilde{\mathfrak{N}}_{*+1}(\Sigma Sl) \to \mathfrak{N}_*(Sl)$. Therefore, the image of $\partial T\colon \mathfrak{N}_*(X) \to \mathfrak{N}_*(\tilde{X})$ is a free $\mathfrak{N}_*$-module with a basis in the image of the homology transfer. It follows that this result holds for the transfer map associated to the double-double cover, $BO_2(\mathbf{H}^\infty) \times S^\infty \times S^\infty \to BO_2(\mathbf{H}^\infty) \times S^\infty \times S^\infty/\mathbf{Z}_2 \times \mathbf{Z}_2$. □

We now need to find a basis for the image of the homology transfer. Consider the fibering $\mathbf{R}P^3 \to \mathbf{R}P(\gamma_{2j+1}^{\mathrm{cx}\,1} \oplus \gamma_n^1 \oplus \gamma_m^1) \to \mathbf{C}P^{2j+1} \times \mathbf{R}P^n \times \mathbf{R}P^m$. $NS^1$ in $S^3$ acts on $\gamma_{2j+1}^{\mathrm{cx}\,1} \to \mathbf{C}P^{2j+1}$ by multiplication, and it acts on $\gamma_n^1 + \gamma_m^1 \to \mathbf{R}P^n \times \mathbf{R}P^m$ trivially. In the induced action on $\mathbf{R}P(\gamma_{2j+1} + \gamma_n + \gamma_m)$, $S^1$ fixes $\mathbf{R}P(\gamma_n + \gamma_m)$, and $\mathbf{Z}_2 \subset S^1$ fixes $\mathbf{R}P(\gamma_{2j+1}) \simeq \mathbf{R}P^{4j+3} \times \mathbf{R}P^n \times \mathbf{R}P^m$, both codimension 2 submanifolds. Thus, we have a $Q_8 \subset NS^1$ action on $\mathbf{R}P(\gamma_{2j+1} + \gamma_n + \gamma_m)$ that induces a branched covering. The normal bundle of $\mathbf{R}P(\gamma_{2j+1})$ in $\mathbf{R}P(\gamma_{2j+1} + \gamma_n + \gamma_m)$ is

$\nu = \mu \otimes (\gamma_n \oplus \gamma_m)$, where $\mu \to \mathbf{R}P^{4j+3} \times \mathbf{R}P^n \times \mathbf{R}P^m$ is the pullback of the standard line bundle over $\mathbf{R}P^{4j+3}$. Then,

$$w(\nu) = (1 + c + a)(1 + c + b) = 1 + (a + b) + (c^2 + c(a + b) + ab),$$

with $a \in H^1(\mathbf{R}P^n; \mathbf{Z}_2)$, $b \in H^1(\mathbf{R}P^m; \mathbf{Z}_2)$, *and* $c \in H^1(\mathbf{R}P^{4j+3}; \mathbf{Z}_2)$, the nonzero classes.

Let $n = 0$; then, $w(\nu) = 1 + b + c(c + b)$, and $u_1^r u_2^{4j+3} = b^r[c^{4j+3}(c + b)^{4j+3}] = c^{4j+3}b^{4j+3+r}$. If $m \geqslant 4j + 3$,

$$u_1^{m-(4j+3)} u_2^{4j+3}\left[\mathbf{R}P(\gamma_{2j+1})\right] = c^{4j+3}b^m[\mathbf{R}P^{4j+3} \times \mathbf{R}P^m] = 1.$$

Since these numbers are the only possible nonzero numbers of the form $u_1^r u_2^s[A]$, we get all possible nonzero numbers $u_1^r u_2^s[A]$ by taking

$$[M] = [\mathbf{R}P(0, m, 2j + 1, 2j + 1)], \quad \text{with } m \geqslant 4j + 3.$$

Because $H^*(BO_2; \mathbf{Z}_2) = \mathbf{Z}_2[w_1, w_2]$, it follows that the homology classes corresponding to the $[\mathbf{R}P(\gamma_{2j+1}), \nu]$ form a basis for the image of

$$\mathrm{tr}\colon H_*(BO_2(\mathbf{H}^\infty) \times S^\infty \times S^\infty/\mathbf{Z}_2 \times \mathbf{Z}_2; \mathbf{Z}_2) \to H_*(BO_2(\mathbf{H}^\infty); \mathbf{Z}_2).$$

PROPOSITION 2.6. *$I_{Q_8}$ in $\mathfrak{N}_*$ is the ideal $I$ generated by* $[\mathbf{R}P(0, m, 2j + 1, 2j + 1)]$, *with* $m \geqslant 4j + 3$.

PROOF. The ideal $I$ is certainly contained in $I_Q$. If $[M] \in I_Q$, then $[M, \mathbf{Z}_2] \in \mathfrak{N}_*(\mathbf{Z}_2)$ is determined by $[F, \nu] \in \mathfrak{N}_{*-2}(BO_2)$, which in turn is determined by $[A, \nu] \in \mathfrak{N}_{*-2}(BO_2)$. By Lemma 2.5, the collection of all such $[A, \nu]$ is a free $\mathfrak{N}_*$-module on a basis given by the homology images of $\nu$: $\mathbf{R}P(\gamma_{2j+1}) \subset \mathbf{R}P(0, m, 2j + 1, 2j + 1)$, $m \geqslant 4j + 3$. Thus, $I_Q = I$. $\square$

Since the above arguments can be used for branchings which come from actions by a generalized quaternion group, we have the following

COROLLARY 2.7. $I_{Q_{2^r}}$, $r \geqslant 3$, is the ideal in $\mathfrak{N}_*$ generated by

$$[\mathbf{R}P(0, m, 2j + 1, 2j + 1)], \qquad m \geqslant 4j + 3. \qquad \square$$

REMARK. If $Q_8$ acts on $M^n$ so that $M \to M/Q$ is a branched covering, then $w_1^s w_i w_j w_k[M^n] = 0$, where $s + i + j + k = n$. These numbers distinguish classes in $\mathfrak{N}_*$ through dimension 7; thus, $(I_Q)_n = 0$ for $n \leqslant 7$. In fact, $(I_Q)_8 = 0$ because $\mathbf{R}P(0, 3, 1, 1)$ bounds.

We find an indecomposable $[\mathbf{R}P(0, 2j + 1, 2j + 1, m)]$, $m \geqslant 4j + 3$, in dimension $n \geqslant 9$ if $n$ is not of the form $2^s - 3$, $2^s - 2$, $2^s - 1$, or $2^s$. Also, if $n$ is of this form, then $[\mathbf{R}P(2j + 1, 2j + 1, n_1, n_2)]$, $4j + 5 + n_1 + n_2 = n$, is decomposable.

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