ON C*-EMBEDDING IN $\beta N$ AND
THE CONTINUUM HYPOTHESIS

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Abstract. Let $\beta N$ denote the Stone-Cech compactification of the natural numbers
$N$ with the discrete topology. It is shown that the continuum hypothesis holds iff for
each pair $X$ and $Y$ of homeomorphic subspaces of $\beta N$, $X$ is C*-embedded in $\beta N$ iff
$Y$ is. Related questions concerning C*-embedded subsets of $\beta N$ are investigated
assuming the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$.

1. Introduction. All hypothesized topological spaces are assumed to be completely
regular and Hausdorff. Thus "space" will mean "completely regular Hausdorff
topological space". For undefined notation and terminology see [GJ or Wa].

Let $S$ be a subspace of a space $X$. In general, the question of whether $S$ is
C*-embedded in $X$ depends not only on the topology of $S$ but also on "how $S$ is
placed in $X". In other words, a space $X$ may contain homeomorphic subspaces $S$
and $T$ with $S$ C*-embedded in $X$ and $T$ not. For example, $Q$ and $Q \setminus \{0\}$ are
homeomorphic dense subspaces of $\beta Q$; the former is C*-embedded in $\beta Q$, the latter
is not ($Q$ denotes the space of rationals and $\beta Q$ its Stone-Cech compactification).
Another example is provided by the homeomorphic subspaces $(-\infty, 0]$ and $(-1, 0]$ of $R$.

The situation can be different when one considers subspaces of $\beta N$. In fact, it is
consistent with the usual axioms of set theory that whether a subspace $X$ of $\beta N$ is
C*-embedded in $\beta N$ depends only on the topology of $X$. Specifically, the following
theorem is 2.2 of [Wo]. Recall that a space $X$ is weakly Lindelöf if, for each open
cover $\mathcal{G}$ of $X$, there exists a countable subfamily $\mathcal{F}$ of $\mathcal{G}$ such that

$$X = \text{cl}_X \left[ \bigcup \{ F : F \in \mathcal{F} \} \right].$$

We denote the continuum hypothesis by CH, and the cardinal $2^{\aleph_0}$ by $c$.

1.1 Theorem. Assume CH. Then the following conditions on a subspace $X$ of $\beta N$ are
equivalent:

(a) $X$ is C*-embedded in $\beta N$.
(b) $|C^*(X)| = c$.
(c) $X$ is weakly Lindelöf.

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However, if one assumes that \( c = 2^{N_1} \) (which also is consistent with the usual axioms of set theory), the situation is known to be different. Denote the discrete space of cardinality \( \alpha \) by \( D(\alpha) \) (thus \( N = D(N_0) \)). The resulting form is due to Efimov (see Remark 8 on p. 274 of [E]).

1.2 Theorem. Assume \( c = 2^{N_1} \). Then \( \beta N \) contains a C*-embedded copy of \( D(N_1) \).

By contrast, Balcar, Simon and Vojtás prove the following result without using any set-theoretic assumptions (see 3.5 of [BSV]). (This result was independently proved (but not published) by K. Kunen and by S. Shelah.)

1.3 Theorem. \( \beta N \) contains a copy \( S \) of \( D(N_1) \) such that \( \text{cl}_{\beta N} S \) is homeomorphic to the one-point compactification of the space \( \{ \alpha \in BD(N_1) : \text{there exists } A \subseteq D(N_1) \text{ such that } |A| \leq N_0 \text{ and } \alpha \in \text{cl}_{BD(N_1)} A \} \). In particular, \( S \) is not C*-embedded in \( \beta N \).

Thus if \( c = 2^{N_1} \), \( \beta N \) contains two homeomorphic subspaces, one C*-embedded in \( \beta N \) and the other not.

1.4 Definition. Let \( \mathcal{P} \) be a topological property.

(a) A space \( X \) has the absolute C*-embedding property for \( \mathcal{P} \) if, whenever \( S \) is a C*-embedded subspace of \( X \), \( S \) has \( \mathcal{P} \), and \( T \) is a subspace of \( X \) that is homeomorphic to \( S \), then \( T \) is C*-embedded in \( X \).

(b) A space \( X \) has the absolute C*-embedding property if \( X \) has the absolute C*-embedding property for \( \mathcal{P} \) for every \( \mathcal{P} \).

Thus \( \beta N \) has the absolute C*-embedding property if CH is assumed, but does not have it if it is assumed that \( c = 2^{N_1} \). This raises the question of whether \( \beta N \) has the absolute C*-embedding property if \( c < 2^{N_1} \). In §2 we show the answer is “no”; in fact, we prove the following, which is the main result of this paper.

1.5 Theorem. The following are equivalent:

(a) CH,

(b) \( \beta N \) has the absolute C*-embedding property.

In §3 we produce examples of some topological properties \( \mathcal{P} \) such that \( \beta N \) has the absolute C*-embedding property for \( \mathcal{P} \) iff \( c < 2^{N_1} \).

2. C*-embedding in \( \beta N \) when CH fails. In this section we prove 1.5. Recall that a space \( X \) is a P-space if its \( G_\delta \)-sets are open. See [GJ or Wa] for basic information on these spaces; note particularly that P-spaces have an open base of clopen sets. Denote by \( \mathcal{B}(X) \) the set of clopen subsets of a space \( X \). The following theorem is implicitly stated and proved in §2 of [DvM]. We include a proof for completeness.

2.1 Theorem. Let \( X \) be a P-space for which \( |\mathcal{B}(X)| \leq c \). Then \( \beta X \) can be embedded in \( \beta N \).

Proof. Note that if \( |\mathcal{B}(X)| \leq c \) then \( |\mathcal{B}(\beta X)| \leq c \). Since \( \beta X \) is zero-dimensional, standard “evaluation map” techniques show that \( \beta X \) can be embedded in \( \{0,1\}^c \), where \( \{0,1\} \) is the two-point discrete space. The argument in §2 of [DvM] then shows that \( \beta X \) can be embedded in the absolute \( E((0,1)^c) \) of \( (0,1)^c \) (see [Wa, Chapter 10, or Wo2] for a discussion of absolutes). Since \( E((0,1)^c) \) is separable and
2.2 Definition. For each ordinal $\alpha$, define $L(\alpha)$ to be the topological space whose underlying set is $\alpha + 1 \setminus \{ \lambda \in \alpha + 1 : \lambda$ is a limit ordinal of countable cofinality$, and which has the subspace topology inherited from the order topology on $\alpha + 1$. (Here, as usual, $\alpha + 1$ is thought of as the set of ordinals no greater than $\alpha$.)

The space $L(\omega_2)$ has been previously used—see [vD or D], for example—to solve problems similar to the ones discussed herein. We collect some known properties of $L(\omega_2)$ in the following.

2.3 Proposition. (a) $L(\alpha)$ is a Lindelöf P-space for every $\alpha$ (the proof is identical to that indicated in [vD] for the case $\alpha = \omega_2$).

(b) Let $T = L(\omega_2) \setminus \{ \omega_2 \}$. Then $T$ is a dense C-embedded subspace of $L(\omega_2)$ and $\nu T = L(\omega_2)$ (9L of [GJ]).

(c) $|\mathcal{B}(T)| = |\mathcal{B}(L(\omega_2))| = c \cdot \aleph_2 [vD]$. 

We need a special case of the following, which is (as indicated below) an immediate consequence of known results.

2.4 Lemma. Let $\alpha$ and $\beta$ be two ordinals. Then $L(\alpha) \times L(\beta)$ is a Lindelöf space.

Proof. If $Y$ is a space, let $Y_8$ denote the space whose underlying set is that of $Y$, and for which the $G_{\delta}$-sets of $Y$ form an open base. It is easy to see that $L(\alpha) \times L(\beta)$ is homeomorphic to $[(\alpha + 1) \times (\beta + 1)]_8$, where $\alpha + 1$ and $\beta + 1$ are given the usual order topology. It is known that if $Y$ is a compact scattered space, then $Y_8$ is Lindelöf; see, for example, p. 27 of [M]. Since $(\alpha + 1) \times (\beta + 1)$ is compact scattered, the lemma follows. □

2.5 Corollary. $L(\omega_2) \times L(\omega_2)$ is Lindelöf and $|\mathcal{B}(L(\omega_2) \times L(\omega_2))| = c \cdot \aleph_2$.

Proof. For the second claim, note that as $L(\omega_2) \times L(\omega_2)$ is Lindelöf, every clopen set of $L(\omega_2) \times L(\omega_2)$ is the union of countably many basic clopen sets of $L(\omega_2) \times L(\omega_2)$ of the form $A \times B$, where $A, B \in \mathcal{B}(L(\omega_2))$. Thus

$$|\mathcal{B}(L(\omega_2) \times L(\omega_2))| \leq \left( |\mathcal{B}(L(\omega_2))| \times |\mathcal{B}(L(\omega_2))| \right)^{\aleph_0}$$

$$= (c \cdot \aleph_2)^{\aleph_0} \quad \text{(follows from 2.3(c))}$$

$$= c \cdot \aleph_2. \quad \square$$

Henceforth we denote the space $L(\omega_2)$ by $L$.

Proof of 1.5. (a) $\Rightarrow$ (b). This is part of 1.1.

(b) $\Rightarrow$ (a). Suppose CH fails. Let $J = T \oplus T$ (the direct sum of two copies of the space $T$ of 2.3(b)). Note that $J$ is a P-space. By 2.3(c), $|\mathcal{B}(J)| = \aleph_2 \cdot c = c$ (as CH fails). Hence, by 2.1, $J$ can be C*-embedded in $\beta N$.

Finite products of P-spaces are P-spaces (4K.6 of [GJ]), so, by 2.5, $L \times L$ is a P-space and $|\mathcal{B}(L \times L)| = c$. Hence, by 2.1, $\beta(L \times L)$ can be embedded in $\beta N$. 

Now $\{(\omega_2, \omega_2) \} \cup (L \times \{ \omega_2 \}) \setminus \{ (\omega_2, \omega_2) \} = J_1$ is homeomorphic to $J$ and is a subspace of $L \times L$. But $(\omega_2, \omega_2)$ is in the $L \times L$ closure of the complementary clopen sets $\{ \omega_2 \} \times L$ and $L \times \{ \omega_2 \}$ of $J_1$, so $J_1$ is not C*-embedded in $L \times L$. Thus
a homeomorph of \( J \) can be embedded in \( \beta N \) in such a way that it is not \( C^* \)-embedded in \( \beta N \). Hence (b) fails. \( \square \)

3. What happens when \( c < 2^{\aleph_1} \). We now show that for certain topological properties \( \mathcal{P}, \beta N \) has the absolute \( C^* \)-embedding property for \( \mathcal{P} \) iff \( c < 2^{\aleph_1} \).

Recall (see [B]) that a space \( X \) is \( \delta\theta \)-refinable if, for each open cover \( \mathscr{C} \) of \( X \), there exists a countable collection \( \{ \gamma_n : n \in \mathbb{N} \} \) of open covers of \( X \), each refining \( \mathscr{C} \), such that for each \( x \in X \) there exists \( n(x) \in \mathbb{N} \) for which \( |\{ V \in \gamma_{n(x)} : x \in V \}| \leq \aleph_0 \). A space is \( \mathcal{N}_1 \)-compact if it has no uncountable closed discrete subsets. The following result of Aull appears in [A1].

3.1 Theorem. An \( \mathcal{N}_1 \)-compact \( \delta\theta \)-refinable space is Lindelöf.

An immediate consequence is the following

3.2 Theorem. The following are equivalent:

(a) \( c < 2^{\aleph_1} \).

(b) \( \beta N \) has the absolute \( C^* \)-embedding property for “normal \( \delta\theta \)-refinable”.

Proof. (a) \( \Rightarrow \) (b). Let \( X \) and \( Y \) be homeomorphic normal \( \delta\theta \)-refinable subspaces of \( \beta N \). If \( X \) is \( \mathcal{N}_1 \)-compact, then, by 3.1, \( X \) is Lindelöf. By 5.2 of [N], Lindelöf subspaces of \( F \)-spaces are \( C^* \)-embedded, so \( X \) and \( Y \) are \( C^* \)-embedded in \( \beta N \). If \( X \) is not \( \mathcal{N}_1 \)-compact, \( X \) contains a closed copy of \( D(\mathcal{N}_1) \). Since \( X \) is normal, this is \( C^* \)-embedded in \( X \). Thus \( |C^*(X)| = |C^*(Y)| \geq |C^*(D(\mathcal{N}_1))| = 2^{\aleph_1} > c = |C^*(\beta N)| \), so neither \( X \) nor \( Y \) is \( C^* \)-embedded in \( \beta N \).

(b) \( \Rightarrow \) (a). If \( c = 2^{\aleph_1} \), then as noted in §1, \( \beta N \) contains two copies of \( D(\mathcal{N}_1) \), one \( C^* \)-embedded and the other not. Since \( D(\mathcal{N}_1) \) is normal \( \delta\theta \)-refinable, (b) fails. \( \square \)

Note that the space \( T \) of 2.3(b) is normal, so “\( \delta\theta \)-refinable” cannot be dropped from 3.2(b) above.

Although not directly connected to our previous work, the following related results are of interest.

3.3 Lemma. Assume \( c < 2^{\aleph_1} \). Then a normal, \( \delta\theta \)-refinable weakly Lindelöf space of weight no greater than \( c \) is Lindelöf.

Proof. It is a special case of 2.4 of [CH] that if \( X \) is weakly Lindelöf, then \( |C^*(X)| \leq w(x)^{\aleph_0} \), where \( w(x) \) denotes the weight of \( X \) (the least cardinality of an open base of \( X \)). Thus \( |C^*(X)| = c \) as \( w(x) \leq c \). Now argue as in the proof of 3.2. \( \square \)

3.4 Example. Assume that \( c = 2^{\aleph_1} \). By 2.1, \( \beta N \setminus N \) contains a \( C^* \)-embedded copy \( S \) of \( D(\mathcal{N}_1) \). It follows that \( N \cup S \) is a normal (in fact, perfectly normal) separable \( \theta \)-refinable space, but is not Lindelöf. Note that \( w(N \cup S) \leq c \) as \( N \cup S \subseteq \beta N \). Thus the assumption “\( c < 2^{\aleph_1} \)” in 3.3 cannot be dropped.

Let \( \mathcal{R}(X) \) denote the collection of regular closed subsets of \( X \).

3.5 Theorem. The following are equivalent:

(a) \( c < 2^{\aleph_1} \).

(b) If \( X \) is normal and \( \delta\theta \)-refinable and \( |\mathcal{R}(X)| \leq c \), then \( X \) is Lindelöf.

(c) Each normal separable \( \theta \)-refinable space is Lindelöf.

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Proof. (a) ⇒ (b). If there were $\aleph_1$ pairwise disjoint nonempty open subsets of $X$, then $|\mathcal{R}(X)| \geq 2^{\aleph_1} > c$. Thus $X$ satisfies the countable chain condition and hence is weakly Lindelöf (see 1.1 of [Wo], for example). Since $X$ is regular, $w(X) \leq |\mathcal{R}(X)|$. Now apply 3.3.

(b) ⇒ (c). If $X$ is separable, then $|\mathcal{R}(X)| \leq c$, since if $T$ is dense in $X$, then $A \rightarrow \text{cl}_X A$ is a bijection from $\mathcal{R}(T)$ onto $\mathcal{R}(X)$.

(c) ⇒ (a). Consider 3.4. □

The authors wish to thank the referee for suggesting Example 3.4 and the inclusion of 3.5(c).

Note that the space $D(\aleph_1)$ witnesses the fact that “weakly Lindelöf” cannot be deleted from the statement of 3.3. The space $T$ of 2.3(b) witnesses that “$\delta\theta$-refinable” cannot be deleted from 3.5(b), while Isbell’s space $\Psi$ (see 51 of [GJ]), which is $\theta$-refinable, witnesses that “normal” cannot be deleted from 3.5(b).

4. Open questions. It seems plausible that a stronger result that 3.2 is true. In particular, it is known (see [Z]) that a normal $\theta$-refinable space must (in the absence of measurable cardinals, which obviously do not concern us here) be realcompact. This together with 3.2 suggest the following

4.1 Question. Assume that $c < 2^\aleph_1$. Does $\beta N$ have the absolute $C^*$-embedding property for realcompactness? In fact, does there exist a realcompact $C^*$-embedded non-weakly-Lindelöf subspace of $\beta N$?

Note that the example $J$ used in the proof of 1.5 is not realcompact. We have the following partial result.

4.2 Theorem. Let $X$ be a subspace of $\beta N$ that is a realcompact $P$-space. The following are equivalent:

(a) Every subspace of $\beta N$ that is homeomorphic to $X$ is $C^*$-embedded in $\beta N$.

(b) $X$ is Lindelöf.

Proof. (b) ⇒ (a). This follows from 5.2 of [N] (as quoted in 3.2).

(a) ⇒ (b). Assume (b) fails. If $|C^*(X)| > c$, then (a) fails and we are done, so assume $|C^*(X)| = c$. Since $X$ is a realcompact $P$-space but is not Lindelöf, it follows from Lemma 3D of [A2] that $X$ contains complementary clopen subsets $A$ and $X \setminus A$, neither of which is Lindelöf. Arguing as in the proof of 5.3 of [DF], we see that there exist compact subsets $K$ and $L$ of $\text{cl}_{\beta X} A \setminus X$ and $\text{cl}_{\beta X} (X \setminus A) \setminus X$, respectively, such that the quotient space $Y$ of $X \cup K \cup L$, formed by collapsing the compact set $K \cup L$ to a point, is a $P$-space containing $X$ as a dense subspace. Then $|C^*(Y)| = c$, so obviously $|\mathcal{R}(Y)| \leq c$. Hence, by 2.1, there exists a copy of $\beta Y$ in $\beta N$. As $\text{cl}_Y A \cap \text{cl}_Y (X \setminus A) \neq \emptyset$, this copy of $\beta Y$ contains a copy of $X$ that is not $C^*$-embedded in $\beta N$. □

We are left with these problems:

4.3 Questions. (a) Suppose that $c < 2^\aleph_1$ and that $X$ is a realcompact $P$-space that is $C^*$-embedded in $\beta N$. Must $X$ be Lindelöf? (Since a weakly Lindelöf $P$-space is easily seen to be Lindelöf, this is a special case of the second question in 4.1.)
(b) Suppose that $X$ is a realcompact space (or even a realcompact $P$-space) and $|f[X]| \leq \aleph_0$ for every $f \in C^*(X)$. Must $X$ be Lindelöf? (See [LR] for work related to this.)

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